χ-ADMISSIBLE SUBALGEBRAS OF $\mathfrak{sl}_{pn}(\mathbb{C})$ AND FINITE W-ALGEBRAS.

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ABSTRACT. Let $\mathfrak{g}$ be a complex simple Lie algebra and e a nilpotent element in $\mathfrak{g}$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing isomorphism and let χ denote the linear function on $\mathfrak{g}$ corresponding to e. To a certain nilpotent subalgebra $\mathfrak{m}$ attached to χ, called a χ-admissible subalgebra of $\mathfrak{g}$, we associate an endomorphism algebra $H(\mathfrak{g}, e, \mathfrak{m})$. When $\mathfrak{m}$ is constructed from a good grading for e, we recover the finite W-algebra associated to e and it is well-known that $\operatorname{gr}_{\mathcal{F}} H(\mathfrak{g}, e, \mathfrak{m})$ is isomorphic to $\mathbb{C}[\mathcal{S}]$ as a graded Poisson algebra where $\mathcal{S}$ is the Slodowy slice of e and $\mathcal{F}$ is the Kazhdan filtration, [15, 7].

In this paper, we consider the case where $\mathfrak{g} = \mathfrak{sl}_{pn}(\mathbb{C})$ and e consists of p Jordan blocks all of the same size n. Here, the only good grading for e is the Dynkin grading and we construct χ-admissible subalgebras non isomorphic to the one derived from this good grading. For these algebras $\mathfrak{m}$, we prove that $\operatorname{gr} H(\mathfrak{g}, e, \mathfrak{m}) \simeq \mathbb{C}[\mathcal{S}]$, generalizing Premet and Gan-Ginzburg's result in this particular case, where $\mathcal{S}$ denotes an analogue to the Slodowy slice.

1. INTRODUCTION

1.1. Let $\mathfrak{g}$ be a complex simple Lie algebra and e a nilpotent element in $\mathfrak{g}$. Since the Killing form $\langle\ ,\ \rangle$ of $\mathfrak{g}$ is non-degenerate, it induces an isomorphism $\kappa : \mathfrak{g} \to \mathfrak{g}^*$ and let $\chi := \kappa(e)$ be the linear function on $\mathfrak{g}$ corresponding to e. By the Jacobson-Morosov Theorem, there exist two elements h and f of $\mathfrak{g}$ such that (e, h, f) forms an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. In particular, the element h is semisimple and its eigenvalues are integers, which defines a $\mathbb{Z}$-grading (called the *Dynkin grading*) $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(h, i)$ where $\mathfrak{g}(h, i) := \{x \in \mathfrak{g}\ ;\ [h, x] = ix\}$. Let

$$\mathfrak{p}_\chi := \bigoplus_{i\geqslant 0} \mathfrak{g}(h, i) \quad \text{and} \quad \mathfrak{m}_\chi := \bigoplus_{i<0} \mathfrak{g}(h, i)$$

be the Jacobson-Morosov parabolic subalgebra of the $\mathfrak{sl}_2$-triple (e, h, f) and $\mathfrak{m}_\chi$ the nilradical of the opposite parabolic $\mathfrak{p}_\chi^-$. Note that both parabolic subalgebras $\mathfrak{p}_\chi$ and $\mathfrak{p}_\chi^-$ are with Levi factor $\mathfrak{g}(h, 0) = \mathfrak{p}_\chi \cap \mathfrak{p}_\chi^-$.

The adjoint group G of $\mathfrak{g}$ acts on $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by the adjoint action and the coadjoint action respectively. For x in $\mathfrak{g}$ (resp. $\mathfrak{g}^*$), denote by $G.x$ the adjoint (resp. coadjoint) orbit of x, $G(x)$ the centralizer of x in G and $\mathfrak{g}^x = \operatorname{Lie} G(x)$ the centralizer of x in $\mathfrak{g}$. It follows from the representation theory of $\mathfrak{sl}_2$ that the restriction to $\mathfrak{g}(h, -1) \times \mathfrak{g}(h, -1)$ of the skew-symmetric bilinear form $(x, y) \mapsto \chi([x, y])$ is non-degenerate and for a Lagrangian subspace $\mathfrak{l}$ of $\mathfrak{g}(h, -1)$ with respect to this form, the nilpotent subalgebra $\mathfrak{m} = \mathfrak{l} \oplus \bigoplus_{j\leqslant -2} \mathfrak{g}(h, j)$ verifies the three following properties:

(A1): $\chi([\mathfrak{m}, \mathfrak{m}]) = \{0\}$;
(A2): $\mathfrak{m} \cap \mathfrak{g}^e = \{0\}$;
(A3): $\dim \mathfrak{m} = (\dim G.e)/2$.

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To the nilpotent element e and the nilpotent subalgebra $\mathfrak{m}$, we associate an endomorphism algebra $H(\mathfrak{g},\chi,\mathfrak{m})$, called *finite W-algebra*, for which we review a construction below, see e.g. [15] or [7]. Finite W-algebras were introduced by Premet in [15]. Before that, in the case where e is an even nilpotent element, (i.e. $\mathfrak{g}(h,1)=\mathfrak{g}(h,-1)=\{0\}$), these algebras already appeared in the Ph.D thesis of Lynch (cf. [14]). The case where e is a principal nilpotent element (i.e. e is regular) was considered by Kostant, [12]. The study of finite W-algebras has attracted lot of mathematicians these recent years, particularly for their importance in the representation theory of $\mathfrak{g}$ as shows up the Skryabin's equivalence [17]. See [13] for a survey on recent developments in this theory.

Let $U(\mathfrak{g})$ and $U(\mathfrak{m})$ be the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{m}$ respectively. By the above property (A1), the linear function χ restricts to a character of $\mathfrak{m}$. The latter extends to a representation $\chi : U(\mathfrak{m}) \to \mathbb{C}$ of $U(\mathfrak{m})$ and we denote by $\mathbb{C}_\chi$ its image. The right multiplication by an element of $\mathfrak{m}$ induces a structure of a right $U(\mathfrak{m})$-module on $U(\mathfrak{g})$ and $\mathbb{C}_\chi$ has a structure of a left $U(\mathfrak{m})$-module obtained from the character χ. Let $I(\mathfrak{g},\chi,\mathfrak{m})$ be the left ideal of $U(\mathfrak{g})$ generated by the elements $x-\chi(x)$, for all $x\in\mathfrak{m}$. Set

$$Q(\mathfrak{g},\chi,\mathfrak{m}) := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \simeq U(\mathfrak{g})/I(\mathfrak{g},\chi,\mathfrak{m}).$$

The adjoint action of $\mathfrak{m}$ on $\mathfrak{g}$ uniquely extends to an action θ of $\mathfrak{m}$ on $U(\mathfrak{g})$. The ideal $I(\mathfrak{g},\chi,\mathfrak{m})$ is stable under this action of $\mathfrak{m}$ because $\theta(m)(x-\chi(x))=[m,x]$ and $\chi([m,x])=0$ for all $m,x\in\mathfrak{m}$. The finite W-algebra $H(\mathfrak{g},\chi,\mathfrak{m})$ is the subspace of all invariant elements of $Q(\chi,\mathfrak{m})$ that is:

$$(1)\, H(\mathfrak{g},\chi,\mathfrak{m}) \;=\; \{u+I(\mathfrak{g},\chi,\mathfrak{m}) \in Q(\mathfrak{g},\chi,\mathfrak{m}) \;;\; \theta(m)(u)\in I(\mathfrak{g},\chi,\mathfrak{m}), \;\text{ for all } m\in\mathfrak{m}\},$$

and we have

$$H(\mathfrak{g},\chi,\mathfrak{m}) \;\simeq\; \mathrm{End}_{U(\mathfrak{g})}\big(U(\mathfrak{g})\otimes_{U(\mathfrak{m})}\mathbb{C}_\chi\big)^{\mathrm{op}}.$$

The algebra structure on $H(\mathfrak{g},\chi,\mathfrak{m})$ is defined by: $(u+I(\mathfrak{g},\chi,\mathfrak{m}))(v+I(\mathfrak{g},\chi,\mathfrak{m})) := uv+I(\mathfrak{g},\chi,\mathfrak{m})$ for $u+I(\mathfrak{g},\chi,\mathfrak{m})$ and $v+I(\mathfrak{g},\chi,\mathfrak{m})$ in $H(\mathfrak{g},\chi,\mathfrak{m})$.

In general, if $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ is a *good grading* for e (i.e. $e\in\mathfrak{g}_2$ and the linear map $\mathrm{ad}\, e : \mathfrak{g}_j \to \mathfrak{g}_{j+2}$ is injective for all $j\leqslant -1$ and surjective for all $j\geqslant -1$) [11] or [1], then the restriction of the skew-symmetric bilinear form $(x,y)\mapsto \chi([x,y])$ to $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$ is non-degenerate and the nilpotent subalgebra $\mathfrak{m}=\mathfrak{l}\oplus\bigoplus_{j\leqslant -2}\mathfrak{g}_j$ verifies the above properties (A1), (A2) and (A3) for all Lagrangian subspace $\mathfrak{l}$ of $\mathfrak{g}_{-1}$. Thus we define an endomorphism algebra $H(\mathfrak{g},\chi,\mathfrak{m})$ by formula ([?]). An important fact is that $H(\mathfrak{g},\chi,\mathfrak{m})$ is independent of the choice of the Lagrangian subspace $\mathfrak{l}$ of $\mathfrak{g}_{-1}$, see [1] (or [7] for the case where the good grading is the Dynkin one). Moreover, Brundan and Goodwin prove that $H(\mathfrak{g},\chi,\mathfrak{m})$ only depends, up to isomorphism, on e, and does not depend on the choice of the good grading for e, [1, Theorem 1].

In this context, we have a natural definition:

Definition 1. *A subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ is called* χ-admissible *if it is nilpotent and it satisfies the above three properties* (A1)*,* (A2) *and* (A3)*.*

In more details, for any nilpotent χ-admissible subalgebra $\mathfrak{m}$, we can define an endomorphism algebra $H(\mathfrak{g},\chi,\mathfrak{m})$ using formula (1) and it is natural to ask if this algebra depends, up to isomorphism, on the choice of the algebra $\mathfrak{m}$. Definition 1 may be viewed as an analogue of [15, Definition 2.3] for a field of characteristic zero (here our base field is $\mathbb{C}$ but the definition makes sense for any field of characteristic zero). In the case where the base field has characteristic $p>0$, Premet proves that the finite W-algebra

associated to a nilpotent element and a χ-admissible subalgebra $\mathfrak{m}$ (within the meaning in [15, Definition 2.3]) does not depend, up to isomorphism, on the choice of the χ-admissible subalgebra $\mathfrak{m}$, [15, Proposition 2.6]. In the case of characteristic zero, the problem remains open.

1.2. In this paper, we consider the case where $\mathfrak{g} = \mathfrak{sl}_{pn}(\mathbb{C})$, with $p, n \in \mathbb{N} \smallsetminus \{0, 1\}$, and where e is a nilpotent element associated to the partition $\underbrace{[n, \ldots, n]}_{p \text{ times}}$ of pn. Thus, e is an even nilpotent element and it follows from [6, Theorem 4.2] that the Dynkin grading is the only good grading for e. We construct in this paper a set Ξ of χ-admissible subalgebras strictly containing $\mathfrak{m}_\chi$ (Theorem 2.7). Precisely, there exists $\mathfrak{m} \in \Xi \smallsetminus \{\mathfrak{m}_\chi\}$ such that $\mathfrak{m}$ and $\mathfrak{m}_\chi$ are not isomorphic as Lie algebras (Proposition 2.13). Thus we construct "new" χ-admissible subalgebras. To know if the algebra $H(\mathfrak{g}, \chi, \mathfrak{m})$, for $\mathfrak{m} \in \Xi$, defined by formula (1) depends on the choice of the subalgebra $\mathfrak{m} \in \Xi$ remains an ambitious problem. Here we begin by proving some interesting properties of the algebra $H(\mathfrak{g}, \chi, \mathfrak{m})$, for $\mathfrak{m} \in \Xi$. When $\mathfrak{m} = \mathfrak{m}_\chi$, the case is well-known and $H(\mathfrak{g}, \chi, \mathfrak{m})$ is the finite W-algebra as considered in [16], see also [2]. It can be described as a quotient of the Yangian $Y_{p,n}$ associated to the Lie algebra $\mathfrak{sl}_{pn}$. Our main result may be summarized as follows (see Theorem 3.4, Proposition 3.8, Theorem 4.7 and Theorem 5.7):

Theorem 1. *Let $\mathfrak{m} \in \Xi$. There exists a filtration $\mathcal{F}$ on the enveloping algebra $U(\mathfrak{g})$ which induces a filtration on the algebra $H(\mathfrak{g}, \chi, \mathfrak{m})$ and there exists a variety $\mathcal{S} \subset \chi + \mathfrak{m}^\circ$, transversal to the coadjoint orbits in $\mathfrak{g}^*$, such that $\mathrm{gr}_\mathcal{F} H(\mathfrak{g}, \chi, \mathfrak{m}) \simeq \mathbb{C}[\mathcal{S}]$ as graded Poisson algebras. (Here, $\mathfrak{m}^\circ$ denotes the annihilator of $\mathfrak{m}$ in $\mathfrak{g}^*$.)*

Moreover, the Skryabin's correspondence holds: we have an equivalence of categories between the abelian category of finitely generated left $U(\mathfrak{g})$-modules on which $\mathfrak{m} - \chi(\mathfrak{m})$ acts locally nilpotently and the category of finitely generated left $H(\mathfrak{g}, \chi, \mathfrak{m})$-modules.

When $\mathfrak{m} = \mathfrak{m}_\chi$, the variety $\mathcal{S}$ is the *Slodowy slice* $\chi + (\mathfrak{g}^e)^*$ and $\mathcal{F}$ is the *Kazhdan filtration* (cf. [7, Section 4]). Theorem 1 extends Premet-Gan-Ginzburg's results in our particular case (cf. [15, Theorem 6.4] and [7, Theorem 4.1]). In order to prove this, we use similar arguments to those of [7]. The principal difficulty is to construct a variety $\mathcal{S}$ that verifies some "good" properties. This is the reason why we restrict ourselves to a particular case.

The rest of this paper is organized as follows.

In Section 2, we construct the set Ξ of certain χ-admissible subalgebras of $\mathfrak{g} = \mathfrak{sl}_{pn}(\mathbb{C})$. From Section 3 till the end of the paper, we fix $\mathfrak{m} \in \Xi$. The variety $\mathcal{S} \subset \chi + \mathfrak{m}^\circ$ is introduced and studied in Section 3. In particular, we prove in this section that $\mathcal{S}$ is transversal to the coadjoint orbits (Theorem 3.4) and that $\mathbb{C}[\mathcal{S}]$ actually admits a natural Poisson structure (induced by the Poisson structure on $\mathbb{C}[\mathfrak{g}^*]$), cf. Proposition 3.8. We define a filtration $\mathcal{F}$ and set up some of its properties in the Section 4. This description leads to Theorem 4.7. Section 5 is devoted to the proof of Theorem 4.7. We also emphasize in this Section that the Skryabin's correspondence holds for any $\mathfrak{m} \in \Xi$ (cf. Theorem 5.7).

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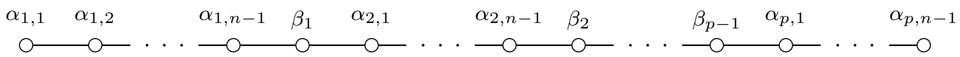
2. χ-admissible subalgebras of $\mathfrak{sl}_{pn}(\mathbb{C})$ in a particular case

Let $(p,n) \in (\mathbb{N} \smallsetminus \{0,1\})^2$. In this section, $\mathfrak{g}$ is the simple Lie algebra $\mathfrak{sl}_{pn}(\mathbb{C})$ consisting of size pn square matrices of trace zero, $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices and $\mathcal{R}$ is the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let $\mathfrak{b}_0$ be the Borel subalgebra of $\mathfrak{g}$ consisting of upper triangular matrices. For all $(i,j) \in \{1,\dots,pn\}^2$, denote by $E_{i,j}$ the square matrix of size pn which has value 1 in the (i,j)-th entry and all other entries are 0 (called an elementary matrix). Let Δ_0 be the set of simple roots associated to $\mathfrak{b}_0$ and $\mathcal{R}^+_{\Delta_0}$ the corresponding set of positive roots. For any $\alpha \in \mathcal{R}$, denote by X_α the unique elementary matrix generating the α-root space $\mathfrak{g}^\alpha$. It will be convenient to index the elements of Δ_0 as follows: For $i \in \{1,\cdots,n-1\}$ and $j \in \{1,\cdots,p\}$, we denote by $\alpha_{j,i}$ the element of $\mathcal{R}$ whose corresponding root space is generated by $E_{(j-1)n+i,(j-1)n+i+1}$ and for $j \in \{1,\cdots,p-1\}$, we denote by β_j the element of $\mathcal{R}$ whose corresponding root space is generated by $E_{jn,jn+1}$. Thus

$$\Delta_0 = \{\alpha_{j,i}\}_{\substack{1\leqslant j\leqslant p \\ 1\leqslant i\leqslant n-1}} \cup \{\beta_1,\dots,\beta_{p-1}\}$$

and the corresponding Dynkin diagram is

(2) 

For a subset P of $\mathcal{R}$, we denote by $\mathfrak{g}^P$ the subspace of $\mathfrak{g}$ generated by the α-root spaces $\mathfrak{g}^\alpha$ where α runs through P. For example, $\mathfrak{h} \oplus \mathfrak{g}^{\mathcal{R}^+_{\Delta_0}} = \mathfrak{b}_0$. Recall that the set of nilpotent orbits of $\mathfrak{sl}_{pn}(\mathbb{C})$ is parametrized by the set of partitions of pn. We consider the $\mathfrak{sl}_2$-triple (e,h,f) of $\mathfrak{sl}_{pn}(\mathbb{C})$,

$$e := \begin{bmatrix} J_n & & 0 \\ & \ddots & \\ 0 & & J_n \end{bmatrix}, \quad h := \begin{bmatrix} D_n & & 0 \\ & \ddots & \\ 0 & & D_n \end{bmatrix}, \quad f := \begin{bmatrix} K_n & & 0 \\ & \ddots & \\ 0 & & K_n \end{bmatrix},$$

where J_n, D_n and K_n are the square matrices of size n given by

$$J_n := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad D_n := \begin{bmatrix} d_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}, \quad K_n := \begin{bmatrix} 0 & & & \\ \mu_1 & \ddots & & \\ & \ddots & \ddots & \\ & & \mu_{n-1} & 0 \end{bmatrix},$$

with $d_i := n+1-2i$ for all $i \in \{1,\dots,n\}$ and $\mu_j := j(n-j)$ for all $j \in \{1,\dots,n-1\}$. We have $h \in \mathfrak{h}$ and $e \in \mathfrak{b}_0$. The nilpotent element e is associated to the partition $[\underbrace{n,\dots,n}_{p \text{ times}}]$.

Remark 2.1. *The elements of $\mathfrak{g}^e$ (resp. $\mathfrak{g}^f$) are the zero trace matrices with p^2 square blocks of size n which are polynomials in J_n (resp. K_n). We deduce that*

$$\dim \mathfrak{g}^e = \dim \mathfrak{g}^f = p^2 n - 1 \quad \textit{and} \quad (\dim G.e)/2 = \frac{p^2 n(n-1)}{2}.$$

Note also that e is an even nilpotent element (i.e. $\mathfrak{g}(h,i) = \{0\}$ for all odd i) and that $\mathfrak{g}(h,0)$ is isomorphic to $(\underbrace{\mathfrak{gl}_p(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_p(\mathbb{C})}_{n \text{ times}}) \cap \mathfrak{sl}_{pn}(\mathbb{C})$. It follows from [6, Theorem 4.2] that $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(h,i)$ is the unique good grading for e. The subalgebra $\mathfrak{m}_\chi :=$

$\bigoplus_{i<0} \mathfrak{g}(h,i)$ is a χ-admissible subalgebra of $\mathfrak{g}$ (cf. Definition 1) because e is even. Our aim in this section is to construct other subalgebras of $\mathfrak{g}$ that are $\mathfrak{h}$-stable (i.e. stable under the adjoint action of $\mathfrak{h}$) and χ-admissible. The intersection $\mathfrak{g}(h,0) \cap \mathfrak{b}_0$ is a Borel subalgebra of $\mathfrak{g}(h,0)$ and

$$\mathfrak{b}^- := (\mathfrak{g}(h,0) \cap \mathfrak{b}_0) \oplus \mathfrak{m}_\chi$$

is a Borel subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}_\chi^-$ and containing $\mathfrak{h}$. We denote by $\mathfrak{n}^-$ its nilradical and by $\mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$ opposite of $\mathfrak{b}^-$. We will search for subalgebras of $\mathfrak{g}$ that are $\mathfrak{h}$-stable and χ-admissible amongst the subalgebras of $\mathfrak{g}$ contained in $\mathfrak{n}^-$. Let Δ be the set of simple roots associated to $\mathfrak{b}$ and $\mathcal{R}_\Delta^+$ the corresponding set of positive roots.

Lemma 2.2. *We have*

$$\Delta := \{\delta_i \,;\, i = 1, \cdots, n-1\} \cup \{\varepsilon_{j,i} \,;\, j = 1, \cdots, p-1,\ i = 1, \cdots n\},$$

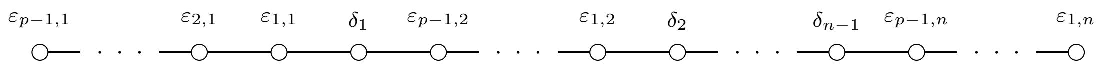
where, for $i = 1, \cdots, n-1$,

$$\delta_i := \alpha_{1,i} + \cdots + \alpha_{1,n-1} + \beta_1 + \alpha_{2,1} + \cdots + \alpha_{p-1,n-1} + \beta_{p-1} + \alpha_{p,1} + \cdots + \alpha_{p,i}$$

and for $j = 1, \cdots, p-1$,

$$\varepsilon_{j,i} = \begin{cases} -\alpha_{j,1} - \cdots - \alpha_{j,n-1} - \beta_j & \textit{if } i = 1; \\ -\beta_j - \alpha_{j+1,1} - \cdots - \alpha_{j+1,n-1} & \textit{if } i = n; \\ -\alpha_{j,i} - \cdots - \alpha_{j,n-1} - \beta_j - \alpha_{j+1,1} - \cdots - \alpha_{j+1,i-1} & \textit{otherwise.} \end{cases}$$

Proof. The set $B = \{\varepsilon_{j,i} \,;\, j = 1, \cdots, p-1,\ i = 1, \cdots n\} \cup \{\delta_i \,;\, i = 1, \cdots, n-1\}$ generates a root subsystem of $\mathcal{R}$ whose simple roots are the elements of B admitting the following Dynkin diagram:



The root spaces relative to this positive root system generate the Borel subalgebra $\mathfrak{b}$. Thus we have $B = \Delta$. ☐

Let $i \in \mathbb{Z} \setminus \{0\}$. Denote by $\mathcal{R}_\Delta(i)$ the subset of $\mathcal{R}$ consisting of roots of height i with respect to the base Δ. This induces a $\mathbb{Z}$-grading on $\mathfrak{g}$ by setting

$$\begin{cases} \mathfrak{g}(\Delta, i) := \mathfrak{g}^{\mathcal{R}_\Delta(i)} & \text{if} \quad i \neq 0; \\ \mathfrak{g}(\Delta, 0) := \mathfrak{h}. \end{cases}$$

Thus,

$$\mathfrak{n}^- = \mathfrak{g}^{-\mathcal{R}_\Delta^+} = \bigoplus_{i=1}^{pn-1} \mathfrak{g}(\Delta, -i).$$

Let h' be the semisimple element of $\mathfrak{g}$ defined by the grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(\Delta, j)$, i.e.

$$\mathfrak{g}(\Delta, j) = \{x \in \mathfrak{g} \mid \operatorname{ad} h'(x) = jx\}, \text{ for all } j \in \mathbb{Z}.$$

By Remark 2.1, notice that $\mathfrak{g}^e$ and $\mathfrak{g}^f$ are $\operatorname{ad} h'$-stable. We represent on Figure 1 the sets $\mathcal{R}_\Delta(i)$ for $p = 4$ and $n = 3$. Precisely, on the square matrix of size 12 of this figure, we have written down i on the (k,l)-th entry if $E_{k,l}$ generates an α-root space $\mathfrak{g}^\alpha$ with α of height i (and 0 if $E_{k,l}$ belongs to $\mathfrak{h}$). The indices corresponding to the roots of Δ are framed. The following lemma can so be easily verified using Lemma 2.2:

$$\left[\begin{array}{ccc@{\qquad}ccc@{\qquad}ccc@{\qquad}ccc}
0 & 4 & 8 & -1 & 3 & 7 & -2 & 2 & 6 & -3 & \boxed{1} & 5 \\
-4 & 0 & 4 & -5 & -1 & 3 & -6 & -2 & 2 & -7 & -3 & \boxed{1} \\
-8 & -4 & 0 & -9 & -5 & -1 & -10 & -6 & -2 & -11 & -7 & -3 \\[2ex]
\boxed{1} & 5 & 9 & 0 & 4 & 8 & -1 & 3 & 7 & -2 & 2 & 6 \\
-3 & \boxed{1} & 5 & -4 & 0 & 4 & -5 & -1 & 3 & -6 & -2 & 2 \\
-7 & -3 & \boxed{1} & -8 & -4 & 0 & -9 & -5 & -1 & -10 & -6 & -2 \\[2ex]
2 & 6 & 10 & \boxed{1} & 5 & 9 & 0 & 4 & 8 & -1 & 3 & 7 \\
-2 & 2 & 6 & -3 & \boxed{1} & 3 & -4 & 0 & 4 & -5 & -1 & 3 \\
-6 & -2 & 2 & -7 & -3 & \boxed{1} & -8 & -4 & 0 & -9 & -5 & -1 \\[2ex]
3 & 7 & 11 & 2 & 6 & 10 & \boxed{1} & 5 & 9 & 0 & 4 & 8 \\
-1 & 3 & 7 & -2 & 2 & 6 & -3 & \boxed{1} & 5 & -4 & 0 & 4 \\
-5 & -1 & 3 & -6 & -2 & 2 & -7 & -3 & \boxed{1} & -8 & -4 & 0
\end{array}\right]$$

FIGURE 1. The sets $\mathcal{R}_\Delta(i)$ for $p = 4$ and $n = 3$

Lemma 2.3. *We have:*
(i) $\mathcal{R}_\Delta(p(n-1)+1) = \{-\beta_j\,;\, j = 1,\cdots,p-1\}$,
(ii) $\mathcal{R}_\Delta(p) = \{\alpha_{j,i}\,;\, j = 1,\cdots p,\ i = 1,\cdots,n-1\} = \operatorname{supp}(e)$ *where* $\operatorname{supp}(e)$ *is the* support *of e, that is the set of roots $\alpha \in \mathcal{R}$ for which the $\mathfrak{g}^\alpha$-component of e in the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathcal{R}} \mathfrak{g}^\alpha$ is non-zero.*

For $r, s \in \{1,\cdots,p\}$, we set

$$\mathbf{E}_{r,s} := \operatorname{span}\big\{E_{k,l}\,;\ (k,l) \in \{(r-1)n+1,\cdots,rn\} \times \{(s-1)n+1,\cdots,sn\}\big\},$$

and

$$\mathcal{R}_{r,s} := \{\alpha \in \mathcal{R}\ ;\ \mathfrak{g}^\alpha \subset \mathbf{E}_{r,s}\}.$$

For $r, s \in \{1,\cdots,p\}$ verifying $r < s$, we set

$$[\![\beta_r,\beta_{s-1}]\!] := \beta_r+\alpha_{r+1,1}+\cdots+\alpha_{r+1,n-1}+\beta_{r+1}+\cdots+\beta_{s-2}+\alpha_{s-1,1}+\cdots+\alpha_{s-1,n-1}+\beta_{s-1},$$

sum of all roots between β_r and β_{s-1} in the diagram (2). Using Lemma 2.2 and Lemma 2.3, the reader may easily verify the following lemma:

Lemma 2.4. *Let $r, s \in \{1,\cdots,p\}$ such that $r < s$.*
(i) *The subset $\mathcal{R}_{r,s}$ is contained in $\mathcal{R}^+_{\Delta_0}$ and we have*

$$\mathcal{R}_{r,s} = \{\alpha \in \mathcal{R};\ [\![\beta_r,\beta_{s-1}]\!] \preccurlyeq \alpha \preccurlyeq \alpha_{r,1}+\cdots+\alpha_{r,n-1}+[\![\beta_r,\beta_{s-1}]\!]+\alpha_{s,1}+\cdots+\alpha_{s,n-1}\},$$

where $\preccurlyeq$ denotes the partial order on $\mathcal{R}$ relative to Δ_0, i.e. for $\alpha, \beta \in \mathcal{R}$, we have $\alpha \preccurlyeq \beta$ if and only if $\beta - \alpha \in \mathcal{R}^+_{\Delta_0}$.
(ii) *For $i, j\ \in \{1,\cdots,n\}$, the elementary matrix $E_{(r-1)n+i,(s-1)n+j}$ generates the root space $\mathfrak{g}^\alpha$ where $\alpha = \alpha_{r,i}+\cdots+\alpha_{r,n-1}+[\![\beta_r,\beta_{s-1}]\!]+\alpha_{s,i}+\cdots+\alpha_{s,j-1}$, and the root α is of height $(r-s)+(j-i)p$ with respect to the base Δ.*

For $i \in \mathbb{Z}\setminus\{0\}$, $1 \leqslant r, s \leqslant p$, we set

$$\mathcal{R}_{r,s}(i) = \mathcal{R}_{r,s} \cap \mathcal{R}_\Delta(i).$$

Lemma 2.5. (i) *For $r, s \in \{1,\cdots,p\}$ such that $r < s$, we have*

$$\mathcal{R}_{r,s}(r-s) = \{-(\varepsilon_{r,k}+\varepsilon_{r+1,k}+\cdots+\varepsilon_{s-1,k})\,;\, k = 1,\cdots,n\}.$$

(ii) *The intersection $\mathfrak{g}^e \cap \mathfrak{g}^{-\mathcal{R}^+_\Delta}$ is generated by the elements,*

$$X^{r,s} := \sum_{\beta \in \mathcal{R}_{r,s}(r-s)} X_\beta,$$

for all $r, s \in \{1, \cdots, p\}$ such that $r < s$.

(iii) *The intersection $\mathfrak{g}^f \cap \mathfrak{g}^{\mathcal{R}_\Delta^+}$ is generated by the elements,*

$$Y^{r,s} := \sum_{\beta \in \mathcal{R}_{r,s}(r-s)} X_{-\beta},$$

for all $r, s \in \{1, \cdots, p\}$ such that $r < s$.

Proof. The assertion (i) follows from Lemma 2.2 and Lemma 2.4(ii). Next, by Remark 2.1 and Lemma 2.4, we deduce (ii) and (iii). □

For any subset $\mathcal{M}$ of $\mathcal{R}$, we set

$$\mathcal{M}_i := \mathcal{M} \cap \mathcal{R}_\Delta(i) \quad (i \neq 0).$$

We say that a subset $\mathcal{M}$ of $\mathcal{R}$ is *closed* if given $\alpha, \beta \in \mathcal{M}$ verifying $\alpha + \beta \in \mathcal{R}$, then $\alpha + \beta \in \mathcal{M}$, cf. e.g. [18, Definition 18.10.1]. If $\mathcal{M}$ is a closed subset of $\mathcal{R}_\Delta^+$, we remark that $\mathfrak{g}^{-\mathcal{M}}$ is a (nilpotent) $\mathfrak{h}$-stable subalgebra of $\mathfrak{g}$ contained in $\mathfrak{n}^-$.

Lemma 2.6. *Let $\mathcal{M}$ be a closed subset of $\mathcal{R}_\Delta^+$.*

(i) *We have $\chi([\mathfrak{g}^{-\mathcal{M}}, \mathfrak{g}^{-\mathcal{M}}]) = \{0\}$ if and only if $(\mathcal{M}_i + \mathcal{M}_{p-i}) \cap \mathcal{R} = \varnothing$ for all $i \in \{1, \cdots, p-1\}$;*

(ii) *The intersection $\mathfrak{g}^e \cap \mathfrak{g}^{-\mathcal{M}}$ is zero if and only if the subset $\mathcal{R}_{r,s}(r-s)$ is not contained in $-\mathcal{M}_{s-r}$ for all $r, s \in \{1, \cdots, p\}$ such that $r < s$.*

Proof. Since $\mathcal{M}$ is contained in $\mathcal{R}_\Delta^+$, we have $\mathfrak{g}^{-\mathcal{M}} = \bigoplus_{i=1}^{pn-1} \mathfrak{g}^{-\mathcal{M}_i}$. Moreover, since $\mathcal{M}$ is closed, we have $(\mathcal{M}_i + \mathcal{M}_j) \cap \mathcal{R} \subseteq \mathcal{M}_{i+j}$ for all $i, j \in \mathbb{Z}$. Hence the assertion (i) is clear because the support of e is $\mathcal{R}_\Delta(p)$ by Lemma 2.3(ii).

The assertion (ii) follows from Remark 2.1 and Lemma 2.5(ii). □

Theorem 2.7. *Let $\mathcal{M}$ be a closed subset of $\mathcal{R}_\Delta^+$. Then, $\mathfrak{g}^{-\mathcal{M}}$ is a χ-admissible subalgebra of $\mathfrak{g}$ if and only if the following conditions hold:*

(C1) *$(\mathcal{M}_i + \mathcal{M}_{p-i}) \cap \mathcal{R} = \varnothing$ for all $i \in \{1, \cdots, p-1\}$;*

(C2) *$\mathcal{R}_{r,s}(r-s) \not\subset -\mathcal{M}_{s-r}$ for all $r, s \in \{1, \cdots, p\}$ such that $r < s$;*

(C3) $\operatorname{card} \mathcal{M}_i + \operatorname{card} \mathcal{M}_{p-i} = p(n-1)$ *for all $i \in \{1, \cdots, p-1\}$ and $\mathcal{M}_i = \mathcal{R}_\Delta(i)$ for all $i \geqslant p$.*

Thus, we have an one-to-one correspondence between the closed subsets $\mathcal{M}$ of $\mathcal{R}_\Delta^+$ verifying conditions (C1), (C2) *and* (C3) *and $\mathfrak{h}$-stable χ-admissible subalgebras of $\mathfrak{g}$ contained in $\mathfrak{n}^-$.*

For example, with

$$\mathcal{M}_\chi := \bigsqcup_{1 \leqslant r < s \leqslant p} \mathcal{R}_{r,s}(p+r-s) \ \sqcup \ \bigsqcup_{i=p}^{pn-1} \mathcal{R}_\Delta(i),$$

we have $\mathfrak{g}^{-\mathcal{M}_\chi} = \mathfrak{m}_\chi$. Before proving Theorem 2.7, we show:

Lemma 2.8. *Let $\mathcal{M}$ be a subset of $\mathcal{R}_\Delta^+$ verifying conditions* (C1) *and* (C2) *of Theorem 2.7. Then for $i \in \{1, \dots, p-1\}$, we have*

$$\operatorname{card} \mathcal{M}_i + \operatorname{card} \mathcal{M}_{p-i} \leqslant p(n-1).$$

Proof. Let $i \in \{1, \dots, p-1\}$. We shall introduce some additional notations. We set

$$T(i) := \mathcal{R}_\Delta(i) \cap (-\mathcal{R}_{\Delta_0}^+), \quad S(i) := \mathcal{R}_\Delta(i) \cap \mathcal{R}_{\Delta_0}^+.$$

We have:

$$T(i) = \bigsqcup_{k=1}^{p-i} T_k(i),$$

where

$$T_k(i) = \{\lambda_{i,k,l} := \varepsilon_{k+i-1,l} + \cdots + \varepsilon_{k,l}\,;\, l = 1, \cdots, n\}.$$

As well, we readily verify that

$$S(i) = \bigsqcup_{k=1}^{i} S_k(i),$$

where

$$S_1(i) = \{\mu_{i,1,l} := \delta_l + \varepsilon_{p-1,l+1} + \cdots + \varepsilon_{p-i+1,l+1}\,;\, l = 1, \cdots, n-1\},$$

and for $k \neq 1$,

$$S_k(i) = \{\mu_{i,k,l} := \varepsilon_{k-1,l} + \cdots + \varepsilon_{1,l} + \delta_l + \varepsilon_{p-1,l+1} + \cdots + \varepsilon_{k+p-i,l+1}\,;\, l = 1, \cdots, n-1\}.$$

We have

$$\operatorname{card} T(i) = n(p-i) \ \text{ and } \ \operatorname{card} S(i) = i(n-1).$$

We have obtained a partition of $\mathcal{R}_\Delta(i)$ given by:

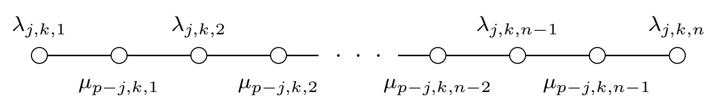
$$\mathcal{R}_\Delta(i) = T(i) \sqcup S(i) = \bigsqcup_{k=1}^{p-i} T_k(i) \sqcup \bigsqcup_{k=1}^{i} S_k(i).$$

For $k \in \{1, \cdots, p-i\}$ and $l \in \{1, \cdots n-1\}$, we have:

- $\lambda_{i,k,l} + \mu_{p-i,k,l} \in \mathcal{R}$ and $\mu_{p-i,k,l} + \lambda_{i,k,l+1} \in \mathcal{R}$;
- For all $\alpha \in \mathcal{R}_\Delta(i) \setminus \{\lambda_{i,k,l}, \lambda_{i,k,l+1}\}$, $\alpha + \mu_{p-i,k,l} \notin \mathcal{R}$.

We deduce that for $k \in \{1, \cdots, p-i\}$ (resp. $k \in \{1, \cdots, i\}$), the roots of $T_k(i) \sqcup S_k(p-i)$ (resp. $T_k(p-i) \sqcup S_k(i)$) form a base of a root system of type $\mathbf{A}_{2n-1}$, whose Dynkin diagram is given by (3):

(3)



where $j = i$ (resp. $j = p-i$). Now let $\mathcal{M}$ be a subset of $\mathcal{R}_\Delta^+$ verifying conditions (C1) and (C2) of Theorem 2.7 and $i \in \{1, \dots, p-1\}$. The defined partitions of $\mathcal{R}_\Delta(i)$ and $\mathcal{R}_\Delta(p-i)$ induce partitions on $\mathcal{M}_i$ and $\mathcal{M}_{p-i}$,

$$(4) \qquad \begin{aligned} \mathcal{M}_i \cup \mathcal{M}_{p-i} &= \bigcup_{k=1}^{p-i} (\mathcal{M}_i \cup \mathcal{M}_{p-i}) \cap (T_k(i) \cup S_k(p-i)) \\ &\quad \cup \bigcup_{k=1}^{i} (\mathcal{M}_i \cup \mathcal{M}_{p-i}) \cap (S_k(i) \cup T_k(p-i)). \end{aligned}$$

Condition (C1) yields that for any $k \in \{1, \ldots, p-i\}$, we have

$$\operatorname{card}\left((\mathcal{M}_i \cup \mathcal{M}_{p-i}) \cap (T_k(i) \cup S_k(p-i))\right) \leqslant n \tag{5}$$

and that for any $k \in \{1, \ldots, i\}$, we have

$$\operatorname{card}\left((\mathcal{M}_i \cup \mathcal{M}_{p-i}) \cap (S_k(i) \cup T_k(p-i))\right) \leqslant n \tag{6}$$

which corresponds to the choice of pairwise orthogonal roots in each diagram (3).
If there exists $k \in \{1, \ldots, p-i\}$ or $k \in \{1, \ldots, i\}$ such that we have an equality in (5) or (6), then condition (C2) is not satisfied by Lemma 2.6(ii) and Lemma 2.5(ii). Therefore, for all $k \in \{1, \ldots, p-i\}$, we have

$$\operatorname{card}\left((\mathcal{M}_i \cup \mathcal{M}_{p-i}) \cap (T_k(i) \cup S_k(p-i))\right) \leqslant n-1,$$

and for all $k \in \{1, \ldots, i\}$, we have

$$\operatorname{card}\left((\mathcal{M}_i \cup \mathcal{M}_{p-i}) \cap (S_k(i) \cup T_k(p-i))\right) \leqslant n-1.$$

By (4) we deduce that

$$\operatorname{card}\mathcal{M}_i + \operatorname{card}\mathcal{M}_{p-i} \leqslant (p-i)(n-1) + i(n-1) = p(n-1).$$

□

Proof of Theorem 2.7. Suppose that $\mathfrak{g}^{-\mathcal{M}}$ is a χ-admissible subalgebra of $\mathfrak{g}$. By properties (A1) and (A2) of Definition 1 and Lemma 2.6, conditions (C1) and (C2) are verified. Let us prove that condition (C3) holds. Let $i \in \{1, \ldots, p-1\}$. By Lemma 2.8, we have

$$\operatorname{card}\mathcal{M}_i + \operatorname{card}\mathcal{M}_{p-i} \leqslant p(n-1).$$

Therefore, we have:

$$\begin{aligned} \dim \mathfrak{g}^{-\mathcal{M}} = \operatorname{card}\mathcal{M} &= \frac{1}{2}\sum_{1 \leqslant i \leqslant p-1} (\operatorname{card}\mathcal{M}_i + \operatorname{card}\mathcal{M}_{p-i}) + \sum_{i=p}^{pn-1} \operatorname{card}\mathcal{M}_i \\ &\leqslant \frac{1}{2}\sum_{1 \leqslant i \leqslant p-1} p(n-1) + \sum_{i=p}^{pn-1} \operatorname{card}\mathcal{R}_\Delta(i) = \frac{p^2 n(n-1)}{2} \end{aligned} \tag{7}$$

because $\operatorname{card}\mathcal{R}_\Delta(i) = pn - i$. Property (A3) of Definition 1 yields that the inequality in (7) is actually an equality. Hence condition (C3) is satisfied.

Conversely, suppose that $\mathcal{M}$ verifies conditions (C1), (C2) and (C3). First, $\mathfrak{g}^{-\mathcal{M}}$ verifies properties (A1) and (A2) by Lemma 2.6. On the other hand, by condition (C3) we see that, in (7), we have an equality. Therefore, property (A3) of Definition 1 is satisfied. At last, since $\mathfrak{g}^{-\mathcal{M}}$ is contained in $\mathfrak{n}^-$, $\mathfrak{g}^{-\mathcal{M}}$ is nilpotent. In conclusion, $\mathfrak{g}^{-\mathcal{M}}$ is χ-admissible and the theorem is proved. □

Remark 2.9. *Suppose $p = 2$. In this case, Theorem 2.7 can be stated in a simplier way: If $\mathcal{M}$ is a closed subset of $\mathcal{R}_\Delta^+$, then the subalgebra $\mathfrak{g}^{-\mathcal{M}}$ of $\mathfrak{n}^-$ is χ-admissible if and only if $\mathcal{M}_1$ contains $n-1$ roots of Δ pairwise orthogonal and $\mathcal{M}_i = \mathcal{R}_\Delta(i)$ for all $i \geqslant 2$. In this case, we notice that $\mathfrak{g}^{-\mathcal{M}}$ is an ideal of $\mathfrak{b}$. Furthermore, the condition "closed" is superfluous.*

Example 2.10. *Let $p = 2$ and $n = 3$. The subsets of Δ consisting of pairwise orthogonal roots are*

$$\{\delta_1, \delta_2\} = \mathcal{M}_\chi,\ \{\delta_1, \varepsilon_3\},\ \{\varepsilon_1, \delta_2\},\ \{\varepsilon_1, \varepsilon_2\},\ \{\varepsilon_2, \varepsilon_3\},\ \{\varepsilon_3, \varepsilon_1\}.$$

For any choice of $\mathcal{M}_1$ *amongst these subsets, the set* $\mathcal{M}_1 \sqcup \bigsqcup_{i\geqslant 2} \mathcal{R}_\Delta(i)$ *is closed. Thus we obtain the* $\mathfrak{h}$*-stable and* χ*-admissible subalgebras of* $\mathfrak{g}$ *contained in* $\mathfrak{n}^-$ *as described below. We denote by* $\mathfrak{m}(\mathcal{M}_1)$ *the subalgebra* $\mathfrak{g}^{-\mathcal{M}}$.

$$\mathfrak{m}(\{\delta_1,\delta_2\}) = \left[\begin{array}{cccccc} 0&0&0&0&0&0\\ \ast&0&0&\ast&0&0\\ \ast&\ast&0&\ast&\ast&0\\ &&&&&\\ 0&0&0&0&0&0\\ \ast&0&0&\ast&0&0\\ \ast&\ast&0&\ast&\ast&0 \end{array}\right], \quad \mathfrak{m}(\{\delta_1,\varepsilon_3\}) = \left[\begin{array}{cccccc} 0&0&0&0&0&0\\ \ast&0&0&\ast&0&0\\ \ast&\ast&0&\ast&\ast&\ast\\ &&&&&\\ 0&0&0&0&0&0\\ \ast&0&0&\ast&0&0\\ \ast&0&0&\ast&\ast&0 \end{array}\right]$$

$$\mathfrak{m}(\{\varepsilon_1,\delta_2\}) = \left[\begin{array}{cccccc} 0&0&0&\ast&0&0\\ \ast&0&0&\ast&0&0\\ \ast&\ast&0&\ast&\ast&0\\ &&&&&\\ 0&0&0&0&0&0\\ 0&0&0&\ast&0&0\\ \ast&\ast&0&\ast&\ast&0 \end{array}\right], \quad \mathfrak{m}(\{\varepsilon_1,\varepsilon_2\}) = \left[\begin{array}{cccccc} 0&0&0&\ast&0&0\\ \ast&0&0&\ast&\ast&0\\ \ast&\ast&0&\ast&\ast&0\\ &&&&&\\ 0&0&0&0&0&0\\ 0&0&0&\ast&0&0\\ \ast&0&0&\ast&\ast&0 \end{array}\right]$$

$$\mathfrak{m}(\{\varepsilon_2,\varepsilon_3\}) = \left[\begin{array}{cccccc} 0&0&0&0&0&0\\ \ast&0&0&\ast&\ast&0\\ \ast&\ast&0&\ast&\ast&\ast\\ &&&&&\\ 0&0&0&0&0&0\\ 0&0&0&\ast&0&0\\ \ast&0&0&\ast&\ast&0 \end{array}\right], \quad \mathfrak{m}(\{\varepsilon_3,\varepsilon_1\}) = \left[\begin{array}{cccccc} 0&0&0&\ast&0&0\\ \ast&0&0&\ast&0&0\\ \ast&\ast&0&\ast&\ast&\ast\\ &&&&&\\ 0&0&0&0&0&0\\ 0&0&0&\ast&0&0\\ \ast&0&0&\ast&\ast&0 \end{array}\right]$$

We end this section by Propositions 2.12 and 2.13 which prove that Theorem 2.7 provides "new" χ-admissible subalgebras. The proof of the following lemma is left to the reader:

Lemma 2.11. *Let* $\mathfrak{a}$ *be a nilpotent* $\mathfrak{h}$*-stable subalgebra of* $\mathfrak{b}$*. Then there exists a closed subset* P *of* $\mathcal{R}_\Delta^+$ *such that* $\mathfrak{a} = \mathfrak{g}^P$ *and we have:*

(i) $\mathfrak{a}^\perp = \mathfrak{b} \oplus \mathfrak{g}^{-Q}$ *where* $\mathfrak{a}^\perp$ *is the orthogonal complement of* $\mathfrak{a}$ *in* $\mathfrak{g}$ *with respect to the Killing form of* $\mathfrak{g}$ *and* $Q = \mathcal{R}_\Delta^+ \setminus P$*;*

(ii) $\mathfrak{a}^\perp$ *is a subalgebra of* $\mathfrak{g}$ *if and only if* $\mathfrak{a}$ *is the nilradical of a parabolic subalgebra of* $\mathfrak{g}$.

Proposition 2.12. *Let* $\mathcal{M}$ *be a closed subset of* $\mathcal{R}_\Delta^+$ *different from* $\mathcal{M}_\chi$ *and verifying conditions* (C1), (C2) *and* (C3) *of Theorem 2.7. Then* $\mathfrak{g}^{-\mathcal{M}}$ *is not the nilradical of a parabolic subalgebra of* $\mathfrak{g}$.

Proof. By Lemma 2.11(i), we have

$$(\mathfrak{g}^{-\mathcal{M}})^\perp = \mathfrak{b}^- \oplus \mathfrak{g}^Q \quad \text{where} \quad Q = \mathcal{R}_\Delta(1) \smallsetminus \mathcal{M}_1 \sqcup \cdots \sqcup \mathcal{R}_\Delta(p-1) \smallsetminus \mathcal{M}_{p-1}.$$

Since $\mathcal{M} \neq \mathcal{M}_\chi$, there exists $i \in \{1, \cdots, p-1\}$ such that $\mathcal{M}_i \cap (-\mathcal{R}_{\Delta_0}^+) \neq \varnothing$. Let us keep the notations of the proof of Lemma 2.8. There exists $k \in \{1, \cdots, p-i\}$ such that $T_k(i) \cap \mathcal{M}_i = T_k(i) \cap \mathcal{M} \neq \varnothing$. Since $\mathcal{M}$ verifies conditions (C1), (C2) and (C3) of Theorem 2.7, we have $\mathrm{card}(T_k(i) \cap \mathcal{M}_i) + \mathrm{card}(S_k(p-i) \cap \mathcal{M}_{p-i}) = n-1$. Thus there exists $j \in \{1, \cdots, n-1\}$ such that either the pair $(\mu_{p-i,k,j}, \lambda_{i,k,j+1})$ does not belong to $\mathcal{M}_{p-i} \times \mathcal{M}_i$, or the pair $(\lambda_{i,k,j}, \mu_{p-i,k,j})$ does not belong to $\mathcal{M}_i \times \mathcal{M}_{p-i}$. But $\mu_{p-i,k,j} + \lambda_{i,k,j+1}$ and $\lambda_{i,k,j} + \mu_{p-i,k,j}$ are roots of $\mathcal{R}$ by the proof of Lemma 2.8 and have height p. Therefore, under the assumptions of the proposition, there exist $\alpha, \beta \in Q$ verifying $\alpha \in \mathcal{R}_\Delta(i) \setminus \mathcal{M}_i$ and $\beta \in \mathcal{R}_\Delta(p-i) \setminus \mathcal{M}_{p-i}$ with $\alpha + \beta \in \mathcal{R}_\Delta(p) = \mathcal{M}_p$. In particular, X_α and X_β belong to $\mathfrak{g}^Q \subset (\mathfrak{g}^{-\mathcal{M}})^\perp$, yet $X_{\alpha+\beta}$ does not belong to $(\mathfrak{g}^{-\mathcal{M}})^\perp$. In other words, $(\mathfrak{g}^{-\mathcal{M}})^\perp$ is not a subalgebra of $\mathfrak{g}$ and the proposition follows by Lemma 2.11(ii). $\square$

Proposition 2.13. *There exists a closed subset $\mathcal{M}$ of $\mathcal{R}_{\Delta}^{+}$ different from $\mathcal{M}_{\chi}$ and verifying conditions* (C1), (C2) *and* (C3) *of Theorem 2.7 such that $\mathfrak{g}^{-\mathcal{M}}$ is not isomorphic to $\mathfrak{g}^{-\mathcal{M}_{\chi}}$ as Lie algebras.*

Proof. Let us keep the notations of the proof of Lemma 2.8. Set $\mathcal{M} := \mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{pn-1}$, where

$$\mathcal{M}_i := \{\lambda_{i,k,l};\ 1 \leqslant k \leqslant p-i,\ 1 \leqslant l \leqslant n-1\} \text{ for } i \in \{1, \cdots, p-1\},$$

and

$$\mathcal{M}_i = \mathcal{R}_{\Delta}(i) \text{ for } i \geqslant p.$$

Then $\mathcal{M}$ verifies all the conditions of Theorem 2.7 and since we are in type **A**, one obtains from a straightforward computation that

$$\begin{aligned}
\dim[\mathfrak{g}^{-\mathcal{M}}, \mathfrak{g}^{-\mathcal{M}}] &= \operatorname{card}\{\, \alpha \in \mathcal{R}_{\Delta}(k)\,;\ 1 \leqslant k \leqslant 2p-1, \mathfrak{g}^{\alpha} \subset [\mathfrak{g}^{-\mathcal{M}}, \mathfrak{g}^{-\mathcal{M}}]\,\} \\
&\quad + \operatorname{card}\{\, \alpha \in \mathcal{R}_{\Delta}(k)\,;\ k \geqslant 2p\,\} \\
&= \frac{1}{2}[p^2 n^2 - (p^2 + 6p - 4)n + 8p - 6].
\end{aligned}$$

On the other hand, from the equality $[\mathfrak{m}_{\chi}, \mathfrak{m}_{\chi}] = \bigoplus_{j \leqslant -4} \mathfrak{g}(h, j)$, we deduce that

$$\dim[\mathfrak{m}_{\chi}, \mathfrak{m}_{\chi}] = \frac{p^2(n-1)(n-2)}{2}.$$

We prove by induction on n that

$$\dim[\mathfrak{g}^{-\mathcal{M}}, \mathfrak{g}^{-\mathcal{M}}] > \dim[\mathfrak{m}_{\chi}, \mathfrak{m}_{\chi}].$$

The proposition is now clear. □

Example 3.1 in the following section gives a subset $\mathcal{M}$ as described in the above proof.

3. An analogue of the Slodowy slice and the Poisson structure

Continue with notations as in the previous section. Let $\mathcal{M}$ be a subset of $\mathcal{R}_{\Delta}^{+}$ verifying conditions (C1), (C2) and (C3) of Theorem 2.7, and set $\mathfrak{m} := \mathfrak{g}^{-\mathcal{M}}$ the corresponding $\mathfrak{h}$-stable and χ-admissible subalgebra of $\mathfrak{g}$ contained in $\mathfrak{n}^{-}$. We construct in this section an affine variety $\mathcal{S}$ having similar properties to those of the Slodowy slice, $\chi + (\mathfrak{g}^{e})^{*}$. Recall by condition (C2) of Theorem 2.7 that the set $\mathcal{R}_{r,s}(r-s)$ is not contained in $-\mathcal{M}_{s-r}$ for all $r, s \in \{1, \cdots, p\}$ such that $r < s$. Then we set

$$S^{r,s} := \sum_{\substack{\beta \in \mathcal{R}_{r,s}(r-s) \\ \beta \notin -\mathcal{M}_{s-r}}} X_{-\beta}.$$

We define an $\operatorname{ad} h$-stable and $\operatorname{ad} h'$-stable subspace $\mathfrak{s}(\mathcal{M})$ of $\mathfrak{g}$ by setting

$$\mathfrak{s}(\mathcal{M}) := \Big(\bigoplus_{j \leqslant 0} \mathfrak{g}(\Delta, j) \cap \mathfrak{g}^{f}\Big) \oplus \sum_{1 \leqslant r < s \leqslant p} \mathbb{C} S^{r,s}. \tag{8}$$

In particular, $\mathfrak{s}(\mathcal{M}) \subset \bigoplus_{i \leqslant p-1} \mathfrak{g}(\Delta, i)$ because for all $r, s \in \{1, \cdots, p\}$ such that $r < s$ we have $s - r \in \{1, \cdots, p-1\}$. When $\mathcal{M} = \mathcal{M}_{\chi}$, note that $\mathfrak{s}(\mathcal{M}) = \mathfrak{g}^{f}$. From now on, the subspace $\mathfrak{s}(\mathcal{M})$ will be simply denoted by $\mathfrak{s}$.

Example 3.1. *Consider the case where $p = 3$ and $n = 3$. We represent on Figure 2 the subalgebra $\mathfrak{m} = \mathfrak{g}^{-\mathcal{M}}$ and the subspace $\mathfrak{s} = \mathfrak{s}(\mathcal{M})$ for $\mathcal{M}_1 = \{\varepsilon_{1,1}, \varepsilon_{1,2}, \varepsilon_{2,1}, \varepsilon_{2,2}\}$, $\mathcal{M}_2 = \{\varepsilon_{2,1} + \varepsilon_{1,1}, \varepsilon_{2,2} + \varepsilon_{1,2}\}$ and $\mathcal{M}_i = \mathcal{R}_{\Delta}(i)$ for $i \geqslant 3$.*

$$\mathfrak{m} = \left[\begin{array}{ccc|ccc|ccc} 0&0&0&*&0&0&*&0&0\\ *&0&0&*&*&0&*&*&0\\ *&*&0&*&*&0&*&*&0\\ \\ 0&0&0&0&0&0&*&0&0\\ 0&0&0&*&0&0&*&*&0\\ *&0&0&*&*&0&*&*&0\\ \\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&*&0&0\\ *&0&0&*&0&0&*&*&0 \end{array}\right], \quad \mathfrak{s} \subset \left[\begin{array}{ccc|ccc|ccc} *&0&0&*&0&0&*&0&0\\ *&*&0&*&*&0&*&*&0\\ *&*&*&*&*&*&*&*&*\\ \\ 0&0&0&*&0&0&*&0&0\\ *&0&0&*&*&0&*&*&0\\ *&*&*&*&*&*&*&*&*\\ \\ 0&0&0&0&0&0&*&0&0\\ *&0&0&*&0&0&*&*&0\\ *&*&*&*&*&*&*&*&* \end{array}\right]$$

FIGURE 2. The subspace $\mathfrak{s} = \mathfrak{s}(\mathcal{M})$ for $p = 3$ and $n = 3$

Lemma 3.2. (i) *We have the inclusions $\mathfrak{s} \subset \mathfrak{m}^\perp$ and $\mathfrak{m}^\perp \subset \bigoplus_{j \leqslant p-1} \mathfrak{g}(\Delta, j)$ where $\mathfrak{m}^\perp$ is the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$.*

(ii) *We have $\mathfrak{g} = [\mathfrak{g}, e] \oplus \mathfrak{s}$ and $\mathfrak{m}^\perp = [\mathfrak{m}, e] \oplus \mathfrak{s}$.*

Proof. (i) By condition (C3) of Theorem 2.7, the subspace $\bigoplus_{j \leqslant -p} \mathfrak{g}(\Delta, j)$ is contained in $\mathfrak{m}$. Therefore, $\mathfrak{m}^\perp \subset \bigoplus_{j \leqslant p-1} \mathfrak{g}(\Delta, j)$. Moreover, the subspace $\mathfrak{s}$ is contained in $\mathfrak{m}^\perp$. In fact, $S^{r,s}$ belongs to $\mathfrak{m}^\perp$ for all (r, s) by construction and $\bigoplus_{j \leqslant 0} \mathfrak{g}(\Delta, j) \subset \mathfrak{m}^\perp$ because $\mathfrak{m}$ is contained in $\bigoplus_{j \leqslant -1} \mathfrak{g}(\Delta, j)$.

(ii) Recall that $\mathfrak{g} = [\mathfrak{g}, e] \oplus \mathfrak{g}^f$. For the first equality, it suffices to show that: $\dim \mathfrak{s} = \dim \mathfrak{g}^f$ and $[\mathfrak{g}, e] \cap \mathfrak{s} = \{0\}$. By using Remark 2.1 and since $\mathfrak{g}^f$ is $\operatorname{ad} h'$-stable, we have,

$$\begin{aligned} \dim\Big(\bigoplus_{j \leqslant 0} \mathfrak{g}(\Delta, j) \cap \mathfrak{g}^f\Big) &= \dim \mathfrak{g}^f - \dim(\mathfrak{g}(\Delta, 1) \cap \mathfrak{g}^f) - \cdots - \dim(\mathfrak{g}(\Delta, p-1) \cap \mathfrak{g}^f) \\ &= \dim \mathfrak{g}^f - (p-1) - (p-2) - \cdots - 1 \;=\; \dim \mathfrak{g}^f - \frac{p(p-1)}{2}. \end{aligned}$$

Therefore, we obtain

$$\dim \mathfrak{s} = \big(\dim \mathfrak{g}^f - \frac{p(p-1)}{2}\big) + \sum_{i=1}^{p-1} (p - i) = \dim \mathfrak{g}^f.$$

Let $x = y + \sum_{1 \leqslant r < s \leqslant p} a_{r,s} S^{r,s}$ be an element of $\mathfrak{s} \cap [\mathfrak{g}, e]$ where $y \in \bigoplus_{j \leqslant 0} \mathfrak{g}(\Delta, j) \cap \mathfrak{g}^f$ and $a_{r,s} \in \mathbb{C}$ for all (r, s). Let us prove that $x = 0$. As $\mathfrak{s}$ and $[\mathfrak{g}, e]$ are $\operatorname{ad} h'$-stable, we can assume that x belongs to $\mathfrak{g}(\Delta, j)$ for some $j \leqslant p - 1$.

- If $j \in \{1, \ldots, p-1\}$, then $x = \sum_{1 \leqslant r < s \leqslant p} a_{r,s} S^{r,s}$. By our hypothese, x belongs to $[\mathfrak{g}, e] = (\mathfrak{g}^e)^\perp$. Recall that $X^{r,s}$ was defined in Lemma 2.5(ii). For all $r, s \in \{1, \ldots, p\}$ verifying $r < s$, we have $\langle x, X^{r,s} \rangle = 0$, which forces $a_{r,s} = 0$ for all (r, s) because $\langle S^{r,s}, X^{r,s} \rangle \neq 0$. Hence $x = 0$.
- If $j \leqslant 0$, then $x \in \mathfrak{g}^f$. Therefore, $x = 0$ because $[\mathfrak{g}, e] \cap \mathfrak{g}^f = \{0\}$.

Let us prove the second equality. By (i), $\mathfrak{s}$ is contained in $\mathfrak{m}^\perp$. Let us show that $\dim \mathfrak{m}^\perp = \dim[\mathfrak{m}, e] + \dim \mathfrak{s}$. As e is even, we have $\dim \mathfrak{g}^f = \dim \mathfrak{g}(h, 0)$. On the other hand, by properties (A2) and (A3) of Definition 1, we have $\dim \mathfrak{m} = \dim \mathfrak{m}_\chi = \dim[\mathfrak{m}, e]$

and $\dim \mathfrak{m}_\chi^\perp = \dim \mathfrak{m}^\perp$. But

$$\dim \mathfrak{m}_\chi^\perp = \dim \bigoplus_{i \leqslant 0} \mathfrak{g}(h,i) = \dim \mathfrak{m}_\chi + \dim \mathfrak{g}^f.$$

Hence we obtain

$$\dim \mathfrak{m}^\perp = \dim \mathfrak{m} + \dim \mathfrak{g}^f = \dim [\mathfrak{m}, e] + \dim \mathfrak{g}^f = \dim [\mathfrak{m}, e] + \dim \mathfrak{s},$$

since $\dim \mathfrak{s} = \dim \mathfrak{g}^f$. From what foregoes, we have $\mathfrak{s} \cap [\mathfrak{g}, e] = \{0\}$. In particular, $\mathfrak{s} \cap [\mathfrak{m}, e] = \{0\}$. So, $\mathfrak{m}^\perp = [\mathfrak{m}, e] \oplus \mathfrak{s}$. $\square$

Recall that the Killing form of $\mathfrak{g}$ induces an isomorphism $\kappa : \mathfrak{g} \to \mathfrak{g}^*, x \mapsto \langle x, \cdot \rangle$ and that the adjoint group G of $\mathfrak{g}$ acts on $\mathfrak{g}$ and $\mathfrak{g}^*$ by the adjoint action and the coadjoint action respectively. We set

$$\mathcal{S} := \chi + \kappa(\mathfrak{s}) \quad \subset \quad \mathfrak{g}^*.$$

The article of Gan-Ginzburg [7] concerns the case where $\mathfrak{s} = \mathfrak{g}^f$ (i.e. $\mathfrak{m} = \mathfrak{m}_\chi$). Here, we shall study properties of the variety $\mathcal{S}$ following the ideas of [7, Section 2]. Let $\gamma : \mathbb{C}^* \to G$ be the one-parameter subgroup generated by $\operatorname{ad} h'$. Thus, for all $j \in \mathbb{Z}$, we have

$$\mathfrak{g}(\Delta, j) = \{x \in \mathfrak{g} \ ; \ \gamma(t)(x) = t^j x, \text{ for all } t \in \mathbb{C}^*\}.$$

We define an action ρ of $\mathbb{C}^*$ on $\mathfrak{g}$ by setting, for all $t \in \mathbb{C}^*$ and all $x \in \mathfrak{g}$:

$$\rho(t)(x) = t^p \gamma(t^{-1})(x).$$

For $x \in \mathfrak{g}(\Delta, j)$ and $t \in \mathbb{C}^*$, we have $\rho(t)(x) = t^{-j+p} x$. In particular, since $e \in \mathfrak{g}(\Delta, p)$, we have $\rho(t)(e) = e$.

Lemma 3.3. *The action ρ stabilizes $e + \mathfrak{s}$ and $e + \mathfrak{m}^\perp$ and it is contracting on these two varieties, i.e.* $\lim\limits_{t \to 0} \rho(t)(e + x) = e$ *for all* $x \in \mathfrak{m}^\perp$.

Proof. Since $\mathfrak{s}$ and $\mathfrak{m}^\perp$ are $\operatorname{ad} h'$-stable, ρ stabilizes $e + \mathfrak{s}$ and $e + \mathfrak{m}^\perp$. Moreover, by Lemma 3.2(i), we have $\lim\limits_{t \to 0} \rho(t)(e + x) = e$ for all $x \in \mathfrak{m}^\perp$. $\square$

Theorem 3.4. *The affine variety $\mathcal{S} = \chi + \kappa(\mathfrak{s})$ (resp. $\chi + \mathfrak{m}^\circ$) is transversal to the coadjoint orbits of $\mathfrak{g}^*$, where $\mathfrak{m}^\circ$ is the annihilator of $\mathfrak{m}$ in $\mathfrak{g}^*$. Precisely, for all $\xi \in \mathcal{S}$ (resp. $\xi \in \chi + \mathfrak{m}^\circ$), we have $T_\xi(G.\xi) + T_\xi(\mathcal{S}) = \mathfrak{g}^*$ (resp. $T_\xi(G.\xi) + T_\xi(\chi + \mathfrak{m}^\circ) = \mathfrak{g}^*$).*

Proof. We will prove the theorem for $\mathcal{S}$. The same arguments apply for $\mathfrak{m}^\circ$ by Lemma 3.2 and Lemma 3.3.

Identify $\mathfrak{g}$ to $\mathfrak{g}^*$ via the isomorphism κ. For all $x \in e + \mathfrak{s}$, we have $\mathrm{T}_x(G.x) = [\mathfrak{g}, x]$ and $\mathrm{T}_x(e + \mathfrak{s}) = \mathfrak{s}$. So it suffices to show that, for all $x \in e + \mathfrak{s}$, $[\mathfrak{g}, x] + \mathfrak{s} = \mathfrak{g}$. Let us fix $x \in e + \mathfrak{s}$.

Let $\eta : G \times (e + \mathfrak{s}) \to \mathfrak{g}$ be the adjoint action map. For all $(g, X) \in G \times (e + \mathfrak{s})$, $v \in \mathrm{T}_g G$ and $w \in \mathfrak{s}$, the differential map of η at (g, X) is given by (cf. e.g. [18, Proposition 29.1.4]):

$$\mathrm{d}\eta_{(g,X)}(v, w) = g([v, X]) + g(w). \tag{9}$$

Thus $\mathrm{d}\eta_{(\mathrm{id},e)}(v, w) = [v, e] + w$. We deduce that the map $\mathrm{d}\eta_{(\mathrm{id},e)}$ is surjective because $[\mathfrak{g}, e] + \mathfrak{s} = \mathfrak{g}$ (cf. Lemma 3.2(ii)). Therefore, $\mathrm{d}\eta_{(\mathrm{id},X)}$ is surjective for all X in some open neighborhood V of e in $e + \mathfrak{s}$. Since the morphism η is G-equivariant for the action given by $a.(b, x) = (ab, x)$, we deduce that the map $\mathrm{d}\eta_{(a,X)}$ is surjective for all $X \in V$ and $a \in G$. By (9), we obtain

$$\mathfrak{g} = [\mathfrak{g}, X] + \mathfrak{s}$$

for all $X \in V$. Set $Y := \{\rho(t)(x)\ ;\ t \in \mathbb{C}^*\}$ and denote by Z the closure of Y in $e + \mathfrak{s}$. Since the $\mathbb{C}^*$-action on $e + \mathfrak{s}$ is contracting, e belongs to Z. Thus $V \cap Y \neq \varnothing$.

Let $X \in V$ and $t \in \mathbb{C}^*$ such that $X = \rho(t)(x)$. Since $\gamma(t^{-1})$ is a Lie automorphism of $\mathfrak{g}$, we have

$$\begin{aligned}[][\mathfrak{g}, X] = [\mathfrak{g}, \rho(t)(x)] = [\mathfrak{g}, t^p\gamma(t^{-1})(x)] = t^p[\mathfrak{g}, \gamma(t^{-1})(x)] \\ = t^p[\gamma(t^{-1})(\mathfrak{g}), \gamma(t^{-1})(x)] = t^p\gamma(t^{-1})([\mathfrak{g}, x]) = \rho(t)([\mathfrak{g}, x]).\end{aligned}$$

We have obtained $\mathfrak{g} = [\mathfrak{g}, X] + \mathfrak{s} = \rho(t)([\mathfrak{g}, x] + \mathfrak{s})$ because $\rho(t)(\mathfrak{s}) = \mathfrak{s}$, whence $\mathfrak{g} = [\mathfrak{g}, x] + \mathfrak{s}$ which completes the proof of the theorem. $\square$

Let M be the unipotent subgroup of G with Lie algebra $\mathfrak{m}$.

Lemma 3.5. *The image of the adjoint map $M \times (e + \mathfrak{m}^{\perp}) \to \mathfrak{g}$ is contained in $e + \mathfrak{m}^{\perp}$.*

Proof. Since $\mathfrak{m}$ is a nilpotent subalgebra, the group M is generated by the elements $\exp \operatorname{ad} m$ where m runs through $\mathfrak{m}$. It suffices to show that for all $m \in \mathfrak{m}$ and all $x \in \mathfrak{m}^{\perp}$, $\exp(\operatorname{ad} m)(e + x)$ belongs to $e + \mathfrak{m}^{\perp}$. Let $m \in \mathfrak{m}$ and $x \in \mathfrak{m}^{\perp}$. We have

$$\exp(\operatorname{ad} m)(e + x) = e + x + [m, e + x] + \cdots + \frac{1}{k!}(\operatorname{ad} m)^k(e + x)$$

for k big enough because $\operatorname{ad} m$ is nilpotent. For all $i \in \mathbb{N}$, $(\operatorname{ad} m)^i(e) \in \mathfrak{m}^{\perp}$ by condition (A1) of Definition 1. On the other hand, since $x \in \mathfrak{m}^{\perp}$, we have $\langle m', [m, x]\rangle = 0$ for all $m' \in \mathfrak{m}$, whence $(\operatorname{ad} m)^i(x) \in \mathfrak{m}^{\perp}$ for all $i \in \mathbb{N}$. Thus, $\exp(\operatorname{ad} m)(e + x) \in e + \mathfrak{m}^{\perp}$ and the lemma is proved. $\square$

By Lemma 3.5, by restriction we define the map,

$$\alpha\ :\ M \times (e + \mathfrak{s}) \longrightarrow e + \mathfrak{m}^{\perp}.$$

We define a $\mathbb{C}^*$-action on $M \times (e + \mathfrak{s})$ by setting:

$$t.(g, x) := \big(\gamma(t^{-1})g\gamma(t), \rho(t)(x)\big),$$

for all $t \in \mathbb{C}^*$, $g \in M$ and $x \in e+\mathfrak{s}$. The action is well defined since $\gamma(t^{-1})(\exp \operatorname{ad} m)\gamma(t) = \exp \operatorname{ad}(\gamma(t^{-1})(m))$ belongs to M for all $m \in \mathfrak{m}$.

Lemma 3.6. (i) *For all $(g, x) \in M \times (e + \mathfrak{s})$, we have: $\lim\limits_{t\to 0} t.(g, x) = (\mathrm{id}, e)$.*

(ii) *The morphism α is $\mathbb{C}^*$-equivariant.*

Proof. (i) Since the $\mathbb{C}^*$-action on $e + \mathfrak{s}$ is contracting (cf. Lemma 3.3), it suffices to show that $\gamma(t^{-1})(\exp \operatorname{ad} m)\gamma(t) \xrightarrow[t\to 0]{} \mathrm{id}$ for all $m \in \mathfrak{m}$. Let $m \in \mathfrak{m}$. By Lemma 3.2(i), we have $\gamma(t^{-1})(m) \xrightarrow[t\to 0]{} 0$, whence

$$\gamma(t^{-1})(\exp \operatorname{ad} m)\gamma(t) = \exp \operatorname{ad}(\gamma(t^{-1})(m)) \xrightarrow[t\to 0]{} \exp(0) = \mathrm{id}\,.$$

(ii) For $t \in \mathbb{C}^*$, $g \in G$ and $x \in e + \mathfrak{m}^{\perp}$, we have:

$$\begin{aligned} \alpha(t.(g, x)) &= \alpha\big(\gamma(t^{-1})g\gamma(t), \rho(t)(x)\big) = \gamma(t^{-1})g\gamma(t)(\rho(t)(x)) \\ &= \gamma(t^{-1})g\gamma(t)\big(t^p\gamma(t^{-1})(x)\big) = t^p\gamma(t^{-1})(g(x)) = \rho(t)(g(x)) = t.\alpha(g, x). \end{aligned}$$

$\square$

Theorem 3.7. *The map*

$$\alpha\ :\ M \times (e + \mathfrak{s}) \longrightarrow e + \mathfrak{m}^{\perp}$$

is an isomorphism of affine varieties.

Proof. Recall the following general result stated in [7, Proof of Lemma 2.1]:

An equivariant morphism $\beta : X_1 \to X_2$ of smooth affine $\mathbb{C}^*$-varieties with contracting $\mathbb{C}^*$-actions which induces an isomorphism between the tangent spaces of the $\mathbb{C}^*$-fixed points must be an isomorphism.

By Lemma 3.6, it suffices to show that the differential of α at (id, e) induces an isomorphism between the tangent space $\mathrm{T}_{(\mathrm{id},e)}(M \times (e+\mathfrak{s})) = \mathfrak{m} \times \mathfrak{s}$ of $M \times (e+\mathfrak{s})$ at (id, e) and the tangent space $\mathrm{T}_e(e+\mathfrak{m}^\perp) = \mathfrak{m}^\perp$ of $e+\mathfrak{m}^\perp$ at e. In fact, the sets of $\mathbb{C}^*$-fixed points of $M \times (e+\mathfrak{s})$ and $e+\mathfrak{m}^\perp$ are $\{(\mathrm{id}, e)\}$ and $\{e\}$ respectively.

We have $\mathrm{d}\alpha_{(\mathrm{id},e)}(\mathfrak{m} \times \mathfrak{s}) = [\mathfrak{m}, e] + \mathfrak{s}$ by the computation as in the proof of Theorem 3.4 and, by Lemma 3.2(ii), $[\mathfrak{m}, e] + \mathfrak{s} = \mathfrak{m}^\perp$. Hence the map $\mathrm{d}\alpha_{(\mathrm{id},e)}$ is surjective. Therefore it is an isomorphism between $\mathfrak{m} \times \mathfrak{s}$ and $\mathfrak{m}^\perp$ for dimension reasons and the theorem follows. $\square$

Recall that the algebra $\mathfrak{g}^*$ has a natural structure of Poisson variety given by

$$\{F_1, F_2\}(\xi) := \xi([\mathrm{d}F_1(\xi), \mathrm{d}F_2(\xi)]),$$

for $F_1, F_2 \in \mathbb{C}[\mathfrak{g}^*]$ and $\xi \in \mathfrak{g}^*$ where, for $i = 1, 2$, $\mathrm{d}F_i(\xi)$ denotes the differential of F_i at ξ. The *symplectic leaves* of $\mathfrak{g}^*$ are the coadjoint orbits of $\mathfrak{g}^*$, [19, Proposition 3.1]. In particular, the coadjoint orbits form a disjoint union of $\mathfrak{g}^*$ and each orbit has a symplectic variety structure.

Proposition 3.8. *The variety $\mathcal{S} \subset \mathfrak{g}^*$ inherits a Poisson structure from that of $\mathfrak{g}^*$.*

Proof. By [19, Proposition 3.10 and Remark 3.11], it suffices to show that:

$$\#\mathrm{Ann}(\mathrm{T}_\xi\mathcal{S}) \cap T_\xi(\mathcal{S}) = \{0\},$$

where $\mathrm{Ann}(\mathrm{T}_\xi\mathcal{S})$ is the annihilator of $T_\xi\mathcal{S} \simeq \kappa(\mathfrak{s})$ in $(\mathrm{T}_\xi\mathfrak{g}^*)^* \simeq (\mathfrak{g}^*)^* \simeq \mathfrak{g}$ and

$$\#_\xi \,:\, T_\xi^*\mathfrak{g}^* \simeq \mathfrak{g} \longrightarrow \mathrm{T}_\xi\mathfrak{g}^* \simeq \mathfrak{g}^*,\ \alpha \longmapsto \xi([\alpha, \cdot]).$$

In fact, by Theorem 3.4, the condition (i) of [19, Remark 3.11] is satisfied. We have

$$\mathrm{Ann}(\mathrm{T}_\xi\mathcal{S}) = \{x \in \mathfrak{g}\ ;\ \eta(x) = 0,\ \text{for all } \eta \in \kappa(\mathfrak{s})\} = \mathfrak{s}^\perp.$$

Therefore,

$$\#\mathrm{Ann}(\mathrm{T}_\xi\mathcal{S}) = \langle\, \kappa^{-1}(\xi), [\mathfrak{s}^\perp, \cdot]\,\rangle = \langle\, [\kappa^{-1}(\xi), \mathfrak{s}^\perp], \cdot\,\rangle = \kappa\big([\kappa^{-1}(\xi), \mathfrak{s}^\perp]\big).$$

We are led to check that the intersection,

$$\kappa([\kappa^{-1}(\xi), \mathfrak{s}^\perp]) \cap T_\xi(\mathcal{S}) = \kappa([\kappa^{-1}(\xi), \mathfrak{s}^\perp]) \cap \kappa(\mathfrak{s}) = \kappa\big([\kappa^{-1}(\xi), \mathfrak{s}^\perp]) \cap \mathfrak{s}\big),$$

is zero. So, the proposition follows from Lemma 3.9 below. $\square$

Lemma 3.9. *Let $\xi \in \mathcal{S}$. Then $[\kappa^{-1}(\xi), \mathfrak{s}^\perp] \cap \mathfrak{s} = \{0\}$.*

Proof. Let $Y \subset e+\mathfrak{s}$ be the set of $y \in e+\mathfrak{s}$ verifying $[y, \mathfrak{s}^\perp] \cap \mathfrak{s} \neq \{0\}$. As $\mathfrak{s}$ and $\mathfrak{s}^\perp$ are $\mathrm{ad}\, h'$-stable, we have for all $t \in \mathbb{C}^*$, $\rho(t)\big([y, \mathfrak{s}^\perp] \cap \mathfrak{s}\big) = [\rho(t)y, \mathfrak{s}^\perp] \cap \mathfrak{s}$. Therefore, ρ stabilizes Y. On the other hand, by Lemma 3.2, e belongs to $(e+\mathfrak{s}) \smallsetminus Y$. Hence at each point y' in some open neighborhood of e in $e+\mathfrak{s}$, we have $y' \in (e+\mathfrak{s}) \smallsetminus Y$.

Suppose that $Y \neq \varnothing$ and let $y \in Y$. Since ρ stabilizes Y, we have $\rho(t)y \in Y$ for all $t \in \mathbb{C}^*$. But for t small enough, $\rho(t)y$ belongs to the open neighborhood V of e, contradiction. $\square$

Remark 3.10. *The latter proof follows the strategy in* [7, §3.1]. *In* [7, §3.2], *Gan and Ginzburg give an alternative proof of Proposition 3.8 (for $\mathfrak{s} = \mathfrak{g}^f$). The arguments used in* [7, §3.2] *can be applied in our cases. In fact, the hypotheses of* [19, Theorem 7.31] *are satisfied by Theorems 3.4 et 3.7. In particular, the Poisson bracket on $\mathbb{C}[\mathcal{S}]$ may be described as follows. Let*

$$\eta : \chi + \mathfrak{m}^\circ \twoheadrightarrow (\chi + \mathfrak{m}^\circ)/M \simeq \mathcal{S}$$

be the projection map given by the coadjoint action of M (Theorem 3.7). Let $F_1, F_2 \in \mathbb{C}[\mathcal{S}]$ and $\widetilde{F}_1, \widetilde{F}_2$ arbitrary extensions of $F_1 \circ \eta \in \mathbb{C}[\chi + \mathfrak{m}^\circ]$ and $F_2 \circ \eta \in \mathbb{C}[\chi + \mathfrak{m}^\circ]$ to $\mathfrak{g}^*$ respectively. Then, if ι denotes the inclusion $\chi + \mathfrak{m}^\circ \hookrightarrow \mathfrak{g}^*$, we have:

$$\{F_1, F_2\} \circ \eta = \{\widetilde{F}_1, \widetilde{F}_2\} \circ \iota.$$

4. An analogue of the Kazhdan filtration

As in previous sections, we fix a closed subset $\mathcal{M}$ of $\mathcal{R}_\Delta^+$ verifying conditions (C1), (C2) and (C3) of Theorem 2.7. Let $\mathfrak{m} := \mathfrak{g}^{-\mathcal{M}}$ be the corresponding $\mathfrak{h}$-stable and χ-admissible subalgebra of $\mathfrak{g}$ contained in $\mathfrak{n}^-$, $\mathfrak{s} = \mathfrak{s}(\mathcal{M})$ be the subspace of $\mathfrak{m}^\perp$ given by formula (8) and $\mathcal{S} = \chi + \kappa(\mathfrak{s})$.

We define the algebra $H(\mathfrak{g}, \chi, \mathfrak{m})$ by formula (1) as explained in the Introduction. It will be simplier denoted by H. In the same way, we denote by I and Q the ideal $I(\mathfrak{g}, \chi, \mathfrak{m})$ and the quotient $Q(\mathfrak{g}, \chi, \mathfrak{m})$ respectively. The aim of this section is to prove the existence of an increasing filtration $\mathcal{F}$ on $U(\mathfrak{g})$ that induces a (positive) filtration on H and a graded algebra structure on $\operatorname{gr}_{\mathcal{F}} H$ such that $\operatorname{gr}_{\mathcal{F}} H \simeq \mathbb{C}[\mathcal{S}]$.

Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$ and $S(\mathfrak{g}) = \bigoplus_{j\in\mathbb{Z}} S^j(\mathfrak{g})$ be the standard grading of $S(\mathfrak{g})$ given by the degree of the elements. The adjoint action of $\operatorname{ad} h'$ on $\mathfrak{g}$ uniquely extends to a derivation on $S(\mathfrak{g})$ and for any $i \in \mathbb{Z}$, let $S_i(\mathfrak{g}) := \{x \in S(\mathfrak{g})\ ;\ (\operatorname{ad} h')(x) = ix\}$. For $k \in \mathbb{Z}$, we set

$$S(\mathfrak{g})[k] := \sum_{i+pj=k} S_i(\mathfrak{g}) \cap S^j(\mathfrak{g}).$$

This defines a $\mathbb{Z}$-grading on $S(\mathfrak{g})$. Since the symmetric algebra $S(\mathfrak{g})$ is canonically isomorphic to $\mathbb{C}[\mathfrak{g}^*]$, the algebra of regular functions on $\mathfrak{g}^*$, the grading $\bigoplus_{k\in\mathbb{Z}} S(\mathfrak{g})[k]$ induces a grading on $\mathbb{C}[\mathfrak{g}^*]$ that we describe below, once again following [7, Section 4] for the case where $\mathfrak{m} = \mathfrak{m}_\chi$. We define a linear action $\rho^\sharp$ of $\mathbb{C}^*$ on $\mathfrak{g}^*$ by setting:

$$\rho^\sharp(t)(\xi) := t^{-p}\gamma(t)(\xi), \text{ for all } t \in \mathbb{C}^*,\ \xi \in \mathfrak{g}^*.$$

We have an induced $\mathbb{C}^*$-action on $\mathbb{C}[\mathfrak{g}^*]$ and we set, for $k \in \mathbb{Z}$,

$$\mathbb{C}[\mathfrak{g}^*](k) := \{F \in \mathbb{C}[\mathfrak{g}^*] \mid \rho^\sharp(t)(F) = t^k F, \text{ for all } t \in \mathbb{C}^*\}.$$

Lemma 4.1. *For all $k \in \mathbb{Z}$, the subspace $\mathbb{C}[\mathfrak{g}^*](k)$ of $\mathbb{C}[\mathfrak{g}^*]$ is identified, via the canonical isomorphism $\mathbb{C}[\mathfrak{g}^*] \simeq S(\mathfrak{g})$, to the subspace $S(\mathfrak{g})[k]$ of $S(\mathfrak{g})$.*

Proof. Let $(i,j) \in \mathbb{Z}^2$ and $x \in S_i(\mathfrak{g}) \cap S^j(\mathfrak{g})$ verifying $i + pj = k$. Denote by F_x the element of $\mathbb{C}[\mathfrak{g}^*]$ corresponding to x. One can assume that x is of form $x = x_1 \dots x_j$ with $x_l \in \mathfrak{g}$ for all $l \in \{1, \dots, j\}$. For $\xi \in \mathfrak{g}^*$ and $t \in \mathbb{C}^*$, we have:

$$\rho^\sharp(t)(F_x)(\xi) = \prod_{l=1}^{j} (\rho^\sharp(t^{-1})(\xi))(x_l) = \prod_{l=1}^{j} (t^p\gamma(t^{-1})(\xi))(x_l) = t^{pj}\xi(\gamma(t)(x)).$$

As $x \in S_i(\mathfrak{g})$, we have $\gamma(t)(x) = t^i x$. We deduce that $\rho^\sharp(t)(F_x) = t^{i+pj} F_x$, i.e. $F_x \in \mathbb{C}[\mathfrak{g}^*](i + pj) = \mathbb{C}[\mathfrak{g}^*](k)$. ∎

Lemma 4.2. (i) *For $x \in \mathfrak{g}(\Delta, j)$ and $t \in \mathbb{C}^*$, we have $\rho^\sharp(t)\kappa(x) = t^{j-p}\kappa(x)$. In particular, since $e \in \mathfrak{g}(\Delta, p)$, we have $\rho^\sharp(t)\chi = \chi$.*

(ii) *The subspaces $\kappa(\mathfrak{s})$ and $\mathfrak{m}^\circ$ are $\rho^\sharp$-stable and the $\rho^\sharp$-weights on $\mathfrak{m}^\perp \supset \mathfrak{s}$ are negative integers.*

Proof. (i) For $x \in \mathfrak{g}(\Delta, j)$, we have $\rho^\sharp(t)\kappa(x) = t^{-p}\gamma(t)\kappa(x) = t^{-p}\kappa(\gamma(t)(x)) = t^{j-p}\kappa(x)$.

(ii) The subspaces $\kappa(\mathfrak{s})$ and $\mathfrak{m}^\circ$ are $\rho^\sharp$-stable because $\mathfrak{s}$ and $\mathfrak{m}^\perp$ are $\operatorname{ad} h'$-stable. As $\mathfrak{m}^\perp \subset \bigoplus_{j \leqslant p-1} \mathfrak{g}(\Delta, j)$ (cf. Theorem 2.7(C3)), the weights on $\kappa(\mathfrak{m}^\perp) = \mathfrak{m}^\circ$ are the integers $j - p$, for $j \leqslant p - 1$. Thus, they are negative integers. $\square$

By Lemma 4.2, the map $\rho^\sharp$ induces a $\mathbb{C}^*$-action on $\mathbb{C}[\mathcal{S}] = \mathbb{C}[\chi + \kappa(\mathfrak{s})]$ and on $\mathbb{C}[\chi + \mathfrak{m}^\circ]$ and these two algebras are so endowed with graded algebra structures. Moreover, we have:

Lemma 4.3. *The grading on $\mathbb{C}[\mathcal{S}]$ (resp. on $\mathbb{C}[\chi + \mathfrak{m}^\circ]$) has no negative graded components and $\mathbb{C}[\mathcal{S}](0) \simeq \mathbb{C}$ (resp. $\mathbb{C}[\chi + \mathfrak{m}^\circ](0) \simeq \mathbb{C}$).*

Proof. We begin by proving the lemma for $\mathbb{C}[\mathcal{S}]$. Let us fix a basis $(z_1, \cdots, z_s)$ of $\mathfrak{s}$ where $z_i \in \mathfrak{g}(\Delta, d_i)$. By Lemma 3.2(i), we have $d_i \leqslant p - 1$. Then the elements $(e, z_1, \cdots, z_s)$ are linearly independent. Set $z_0 := e$ and let us complete this set by homogeneous elements for the grading $\mathfrak{g} = \bigoplus\limits_{j \in \mathbb{Z}} \mathfrak{g}(\Delta, j)$ to a basis $(z_0, z_1, \cdots, z_s, z_{s+1}, \cdots z_r)$ of $\mathfrak{g}$. Let $(\varphi_0, \cdots, \varphi_r)$ be the dual basis of $(\kappa(z_0), \cdots, \kappa(z_r))$. Then $\mathbb{C}[\mathfrak{g}^*] = \mathbb{C}[\varphi_0, \cdots, \varphi_r]$ and for all $i \in \{0, \cdots, r\}$, we have

$$\rho^\sharp(t)(\varphi_i)(\sum_{j=0}^{r} a_j \kappa(z_j)) = \varphi_i(t^{-d_i+p} a_i \kappa(z_i)),$$

whence $\rho^\sharp(t)(\varphi_i) = t^{-d_i+p}\varphi_i$. Let $\mathcal{I}(\mathcal{S}) \subset \mathbb{C}[\mathfrak{g}^*]$ be the ideal of $\mathcal{S}$. Then

$$\mathcal{I}(\mathcal{S}) = \{F \in \mathbb{C}[\varphi_0, \cdots, \varphi_r] \ ; \ F(\kappa(e) + \kappa(z_i)) = 0, \text{ for all } i \in \{1, \cdots, s\}\}.$$

The above equality shows that $\mathcal{I}(\mathcal{S})$ is the ideal of $\mathbb{C}[\mathfrak{g}^*]$ generated by the functions $\varphi_0 - 1, \varphi_{s+1} \cdots, \varphi_r$. Then, $\mathbb{C}[\mathcal{S}] = \mathbb{C}[\psi_1, \cdots, \psi_s]$ where ψ_i is the class of φ_i modulo $\mathcal{I}(\mathcal{S})$ for all i. Note that $\mathcal{I}(\mathcal{S})$ is stable under the action $\rho^\sharp$. Therefore, for $i \in \{1, \cdots, s\}$, ψ_i is of $\rho^\sharp$-weight $-d_i + p \geqslant 1$ because $d_i \leqslant p - 1$. We deduce that $\mathbb{C}[\mathcal{S}]$ has positive graded components. Moreover, $F \in \mathbb{C}[\mathcal{S}]$ is of weight zero if and only if F is constant.

The same arguments go for $\chi + \mathfrak{m}^\circ$ since $\mathfrak{m}^\perp \subset \bigoplus_{j \leqslant p-1} \mathfrak{g}(\Delta, j)$ (cf. Lemma 3.2(i)). $\square$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $(U^j(\mathfrak{g}))_j$ be its standard filtration. The adjoint action of h' on $\mathfrak{g}$ uniquely extends to a derivation on $U(\mathfrak{g})$ and we set, for all $i \in \mathbb{Z}$, $U_i(\mathfrak{g}) := \{x \in U(\mathfrak{g}) \ ; \ (\operatorname{ad} h')(x) = ix\}$. Let $\mathcal{F}$ be the increasing filtration of $U(\mathfrak{g})$ defined by,

$$\mathcal{F}_k U(\mathfrak{g}) := \sum_{i+pj \leqslant k} U_i(\mathfrak{g}) \cap U^j(\mathfrak{g}), \quad (k \in \mathbb{Z}).$$

For $(r, s) \in \mathbb{Z}^2$, $x \in U^r(\mathfrak{g})$ (resp. $x \in U_r(\mathfrak{g})$) and $y \in U^s(\mathfrak{g})$ (resp. $y \in U_s(\mathfrak{g})$), notice that $[x, y] \in U^{r+s-1}(\mathfrak{g})$ (resp. $[x, y] \in U_{r+s}(\mathfrak{g})$). If $x \in \mathcal{F}_r U(\mathfrak{g})$ and $y \in \mathcal{F}_s U(\mathfrak{g})$, then we have $[x, y] \in \mathcal{F}_{r+s-p} U(\mathfrak{g})$. By the Poincaré-Birkhoff-Witt Theorem it follows that $\operatorname{gr}_\mathcal{F} U(\mathfrak{g})$ are $S(\mathfrak{g})$ isomorphic as graded algebras. Here, $S(\mathfrak{g})$ is equipped with the grading $(S(\mathfrak{g})[k])_k$ and $\operatorname{gr}_\mathcal{F} U(\mathfrak{g})$ is the graded algebra of $U(\mathfrak{g})$ with respect to the filtration $\mathcal{F}$, i.e. $\operatorname{gr}_\mathcal{F} U(\mathfrak{g}) = \bigoplus_k \operatorname{gr}_{\mathcal{F},k} U(\mathfrak{g})$ where $\operatorname{gr}_{\mathcal{F},k} U(\mathfrak{g}) := \mathcal{F}_k U(\mathfrak{g}) / \mathcal{F}_{k-1} U(\mathfrak{g})$.

Remark 4.4. *Since* $U(\mathfrak{g})$ *is* almost commutative *with respect to the filtration* $\mathcal{F}$ *(cf.* [5, Definition 1.3.1]*), the (commutative) graded algebra* $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g})$ *has a natural Poisson structure. This Poisson structure coincides with the natural Poisson structure on* $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g}) \simeq S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$.

Recall that Q is the quotient $U(\mathfrak{g})/I$. Let $\pi : U(\mathfrak{g}) \to Q$ be the quotient map and set

$$\mathcal{F}_k Q := \pi(\mathcal{F}_k U(\mathfrak{g})), \quad (k \in \mathbb{Z}).$$

This endows Q with the structure of a filtered module over $U(\mathfrak{g})$.
Let $\mathrm{gr}\,\pi : \mathrm{gr}\, U(\mathfrak{g}) \to \mathrm{gr}\, Q$ be the associated graded surjective homomorphism. We have $\mathrm{gr}_{\mathcal{F}} Q \simeq \mathrm{gr}_{\mathcal{F}} U(\mathfrak{g})/ \mathrm{gr}_{\mathcal{F}} I$ (cf. e.g. [18, Proposition 7.5.3]), and in particular, $\mathrm{gr}_{\mathcal{F}} I$ is the kernel of $\mathrm{gr}(\pi)$.

Lemma 4.5. *Via the isomorphism* $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$*, we identify the kernel* $\mathrm{gr}_{\mathcal{F}} I$ *of* $\mathrm{gr}(\pi)$ *to the ideal* $\mathcal{I}(\chi+\mathfrak{m}^\circ)$ *of* $\mathbb{C}[\mathfrak{g}^*]$ *consisting of all polynomial functions on* $\mathfrak{g}^*$ *vanishing on* $\chi + \mathfrak{m}^\circ$.

Proof. Let $(y_1, \cdots, y_m)$ be a basis of $\mathfrak{m}$ that we complete to a basis

$$(y_1, \cdots, y_m, y_{m+1}, \cdots, y_r)$$

of $\mathfrak{g}$. Since $\mathfrak{m}$ is $\mathrm{ad}\, h'$-stable, we can assume that $y_i \in \mathfrak{g}(\Delta, d_i)$, where $d_i \in \mathbb{Z}$ for all $i \in \{1, \cdots, r\}$. Let $i \in \{1, \cdots, r\}$. We set,

$$\tilde{y_i} := y_i - \chi(y_i).$$

By Theorem 2.7(C3), we have $\bigoplus\limits_{j \leqslant -p} \mathfrak{g}(\Delta, j) \subset \mathfrak{m}$. Therefore, for $i > m$, we have $d_i \geqslant -p+1$. If $d_i \neq -p$, then $\chi(y_i) = \langle e, y_i \rangle = 0$ because $e \in \mathfrak{g}(\Delta, p)$. Hence, $\tilde{y_i} = y_i \in U^1(\mathfrak{g}) \cap U_{d_i}(\mathfrak{g}) \subset \mathcal{F}_{p+d_i} U(\mathfrak{g})$. It follows that

$$\chi = \sum_{j=1}^{m} \chi(y_j) y_j^* + \sum_{j=m+1}^{r} \chi(y_j) y_j^* = \sum_{j=1}^{m} \chi(y_j) y_j^*. \tag{10}$$

Furthermore, by (i), $\ker \mathrm{gr}(\pi) = \mathrm{gr}_{\mathcal{F}} I$ is generated by the elements $\tilde{y_i} + \mathcal{F}_{p+d_i-1} U(\mathfrak{g})$ for $i \in \{1, \cdots, m\}$. Let $(y_1^*, \cdots, y_r^*)$ be the dual basis of $(y_1, \ldots, y_r)$. The set $\nu(\mathrm{gr}_{\mathcal{F}} I)$ of the common zeros of $\mathrm{gr}_{\mathcal{F}} I$ is the set of the elements $\psi = \sum\limits_{j=1}^{r} \lambda_j y_j^* \in \mathfrak{g}^*$, with $(\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$, verifying $\psi(\tilde{y_i}) = 0$, for all $i \in \{1, \cdots, m\}$, i.e. $\lambda_i = \chi(y_i)$ for all $i \in \{1, \cdots, m\}$. In other words, since $(y_{m+1}^*, \cdots, y_r^*)$ is a basis of $\mathfrak{m}^\circ$, an element ψ of $\mathfrak{g}^*$ belongs to $\nu(\mathrm{gr}_{\mathcal{F}} I)$ if and only if ψ belongs to $\chi + \mathfrak{m}^\circ$ by (10). Therefore, the ideal $\mathcal{I}(\chi + \mathfrak{m}^\circ)$ of polynomial functions on $\mathfrak{g}^*$ vanishing on $\chi + \mathfrak{m}^\circ$ is equal to $\mathcal{I}(\nu(\mathrm{gr}_{\mathcal{F}} I)) = \sqrt{\mathrm{gr}_{\mathcal{F}} I} = \mathrm{gr}_{\mathcal{F}} I$ because I is the ideal generated by the affine functions $x - \chi(x)$ where $x \in \mathfrak{m}$. □

Proposition 4.6. *We have an* $\mathfrak{m}$*-equivariant isomorphism,*

$$\vartheta \,:\, \mathrm{gr}_{\mathcal{F}} Q \to \mathbb{C}[\chi + \mathfrak{m}^\circ],$$

between the graded algebras $\mathrm{gr}_{\mathcal{F}} Q$ *and* $\mathbb{C}[\chi + \mathfrak{m}^\circ]$.

Proof. Recall that one has the graded algebras isomorphisms: $\mathrm{gr}_{\mathcal{F}} U(\mathfrak{g}) \simeq S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$. By Lemma 4.5, we deduce the following isomorphisms:

$$\mathrm{gr}_{\mathcal{F}} Q \simeq \mathrm{gr}_{\mathcal{F}} U(\mathfrak{g})/\mathrm{gr}_{\mathcal{F}} I \simeq \mathbb{C}[\mathfrak{g}^*]/\mathcal{I}(\chi + \mathfrak{m}^\circ) \simeq \mathbb{C}[\chi + \mathfrak{m}^\circ].$$

It follows that we have a graded algebra isomorphism $\vartheta : \mathrm{gr}_{\mathcal{F}} Q \to \mathbb{C}[\chi + \mathfrak{m}^\circ]$. The canonical isomorphism $S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$ is $\mathfrak{g}$-equivariant. As $\mathrm{gr}_{\mathcal{F}} I \simeq \mathcal{I}(\chi + \mathfrak{m}^\circ)$ is an

$\mathfrak{m}$-equivariant isomorphism, the isomorphism ϑ is also $\mathfrak{m}$-equivariant. At last, by construction of the grading on $\mathbb{C}[\chi+\mathfrak{m}^\circ]$, the isomorphism ϑ clearly preserves the graded algebra structures. $\square$

Since the adjoint action of $\mathfrak{m}$ stabilizes both I and $\mathcal{F}_k U(\mathfrak{g})$ for all k, the filtration $(\mathcal{F}_k Q)_k$ induces a filtration on H via the inclusion $H \hookrightarrow Q$. Note that $\mathrm{gr}_{\mathcal{F}} H \hookrightarrow \mathrm{gr}_{\mathcal{F}} Q$ is an injective homomorphism of graded algebras. Let $\nu : \mathbb{C}[\chi+\mathfrak{m}^\circ] \to \mathbb{C}[\mathcal{S}]$ be the comorphism corresponding to the inclusion $\mathcal{S} \hookrightarrow \chi+\mathfrak{m}^\circ$. By Proposition 4.6, we have a homomorphism of graded algebras $\nu : \mathrm{gr}_{\mathcal{F}} H \to \mathbb{C}[\mathcal{S}]$.

In the next section, we prove the main result of this paper:

Theorem 4.7. *The homomorphism*

$$\nu : \mathrm{gr}_{\mathcal{F}} H \to \mathbb{C}[\mathcal{S}]$$

is an isomorphism of graded Poisson algebras.

5. The proof of the main result

The aim of this section is to prove Theorem 4.7. We will follow the strategy [7, Section 5] that can be readily applied to our case as we will show.

Recall that the adjoint action of $\mathfrak{m}$ on $U(\mathfrak{g})$ induces an action on Q (because $\mathfrak{m}$ stabilizes I). Consider the standard cochain complex of the $\mathfrak{m}$-module Q:

$$0 \to C^0 \to C^1 \to \dots \to C^i \to \dots$$

where $C^i := \mathrm{Hom}(\bigwedge^i \mathfrak{m}, Q) \simeq (\bigwedge^i \mathfrak{m})^* \otimes Q \simeq \bigwedge^i \mathfrak{m}^* \otimes Q$, and let $\partial^i : C^i \to C^{i+1}$ be the corresponding differential maps. For all $j \in \mathbb{Z}$, we set $\mathfrak{m}^*(j) := \{\xi \in \mathfrak{m}^* ;\ (\mathrm{ad}^* h')\xi = j\xi\}$; observe that $\mathfrak{m}^*(j) \simeq (\mathfrak{g}(\Delta,-j)\cap\mathfrak{m})^*$. Let $i \in \mathbb{N}$. Since $\mathfrak{m} \subset \bigoplus\limits_{j\leqslant -1} \mathfrak{g}(\Delta,j)$, we have the inclusion $\mathfrak{m}^* \subset \bigoplus\limits_{j\geqslant 1} \mathfrak{m}^*(j)$ and,

$$\bigwedge^i \mathfrak{m}^* = \bigoplus_{q\geqslant 1} (\bigwedge^i \mathfrak{m}^*)_q, \quad \text{où} \quad (\bigwedge^i \mathfrak{m}^*)_q := \bigoplus_{j_1+\cdots+j_i=q} \mathfrak{m}^*(j_1) \bigwedge \cdots \bigwedge \mathfrak{m}^*(j_i).$$

We define an increasing filtration on C^i by setting:

$$\mathscr{F}_k C^i := \sum_{q+j\leqslant k} (\bigwedge^i \mathfrak{m}^*)_q \otimes \mathcal{F}_j Q, \quad (k \in \mathbb{Z}).$$

Note that $C^0 = Q$ and that $\mathscr{F}_k C^0 = \mathcal{F}_k Q$ for all k. Moreover, $\mathscr{F}_k C^i = 0$ for k small enough.

Lemma 5.1. *The complex $C^\bullet$ is stabilized by the filtration, i.e. $\partial(\mathscr{F}_k C^i) \subset \mathscr{F}_k C^{i+1}$ for all i.*

Proof. Let $f \in \mathscr{F}_k C^i$; let us show that $\partial f \in \mathscr{F}_k C^{i+1}$. We can assume that $f = \varphi \otimes v$ where $\varphi \in (\bigwedge^i \mathfrak{m}^*)_q$ and $v \in \mathcal{F}_j Q$ such that $q + j \leqslant k$. Let $Y_1, \cdots, Y_{i+1}$ be elements of

$\mathfrak{m}$ such that for all $r \in \{1, \cdots, i+1\}$, $Y_r \in \mathfrak{g}(\Delta, \lambda_r)$ with $\lambda_r \leqslant -1$. Then

$$\begin{aligned}
\partial f(Y_1 \wedge \cdots \wedge Y_{i+1}) &= \sum_{l=1}^{i+1} (-1)^l Y_l \cdot f(Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_{i+1}) + \\
&\quad \sum_{1 \leqslant l < m \leqslant i+1} (-1)^{l+m} f([Y_l, Y_m] \wedge Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge \widehat{Y_m} \wedge Y_{i+1}) \\
&= \sum_{l=1}^{i+1} (-1)^l Y_l \cdot (\varphi(Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_{i+1}) v) + \\
&\quad \sum_{1 \leqslant l < m \leqslant i+1} (-1)^{l+m} \varphi([Y_l, Y_m] \wedge Y_1 \wedge \cdots \widehat{Y_l} \cdots \wedge \widehat{Y_m} \wedge \cdots \wedge Y_{i+1}) v
\end{aligned}$$

If $\sum\limits_{r \neq l} \lambda_r \neq -q$, then $\varphi(Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge Y_{i+1})$ is zero. On the other hand, $Y_l v \in \mathcal{F}_{j+\lambda_l} Q$. Similarly, if $\sum\limits_r \lambda_r \neq -q$, then $\varphi([Y_l, Y_m], Y_1 \wedge \cdots \wedge \widehat{Y_l} \wedge \cdots \wedge \widehat{Y_m} \wedge \cdots \wedge Y_{i+1})$ is zero. Thus, $\partial f \in \sum\limits_{\lambda} (\bigwedge^i \mathfrak{m}^*)_{q-\lambda} \otimes \mathcal{F}_{j+\lambda} Q + (\bigwedge^i \mathfrak{m}^*)_q \otimes \mathcal{F}_j Q$, as a result, $\partial f \in \mathscr{F}_k C^{i+1}$. □

Recall that the $\mathfrak{m}$-module structure on Q induces an $\mathfrak{m}$-module structure on $\operatorname{gr}_{\mathcal{F}} Q$. Let us consider the complex,

$$G^i := \operatorname{Hom}(\bigwedge^i \mathfrak{m}, \operatorname{gr}_{\mathcal{F}} Q) \simeq (\bigwedge^i \mathfrak{m})^* \otimes \operatorname{gr}_{\mathcal{F}} Q \simeq \bigwedge^i \mathfrak{m}^* \otimes \operatorname{gr}_{\mathcal{F}} Q,$$

associated to $\operatorname{gr}_{\mathcal{F}} Q$, and denote by δ the corresponding differential map. We have a grading on G^i given by $G^i = \bigoplus\limits_k G^i_k$ where $G^i_k := \bigoplus\limits_{q+j=k} (\bigwedge^i \mathfrak{m}^*)_q \otimes \mathcal{F}^j Q / \mathcal{F}^{j+1} Q$, with $\mathcal{F}^j Q := \mathcal{F}_{-j} Q$. For $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, we set

$$\mathscr{F}^k C^i := \mathscr{F}_{-k} C^i,$$

such that the filtration $(\mathscr{F}^k C^i)_k$ is decreasing and $\mathscr{F}^k C^i = 0$ for i big enough.

Remark 5.2. *Let $(k, i) \in \mathbb{Z} \times \mathbb{N}$.*
1) *We have $G^i_k \simeq \mathscr{F}^k C^i / \mathscr{F}^{k+1} C^i$.*
2) *We have $\delta(G^i_k) \subset G^{i+1}_k$; This follows from Lemma 5.1.*

The inclusion $\mathscr{F}^k(C^i) \hookrightarrow C^i$ induces a homomorphism $H^i(\mathscr{F}^k(C^\bullet)) \to H^i(C^\bullet)$. By setting $\mathscr{F}^k H^i(\mathfrak{m}, Q)$ to be the image of $H^i(\mathscr{F}^k(C^\bullet))$ in $H^i(C^\bullet)$, we define a filtration on $H^i(C^\bullet) = H^i(\mathfrak{m}, Q)$. In other words,

$$\mathscr{F}^k H^i(\mathfrak{m}, Q) \simeq \frac{\ker \partial^i \cap \mathscr{F}^k C^i}{\operatorname{im} \partial^{i-1} \cap \mathscr{F}^k C^i}.$$

We can then consider the associated graded algebra:

$$\operatorname{gr}_{\mathscr{F}} H^i(\mathfrak{m}, Q) = \bigoplus_k \mathscr{F}^k H^i(\mathfrak{m}, Q) / \mathscr{F}^{k+1} H^i(\mathfrak{m}, Q).$$

Remark 5.3. *For $i = 0$, we have $\operatorname{gr}_{\mathscr{F}} H^0(\mathfrak{m}, Q) = \operatorname{gr}_{\mathscr{F}} Q^{\mathfrak{m}} = \operatorname{gr}_{\mathcal{F}} Q^{\mathfrak{m}}$.*

Our next step is to show that

$$\operatorname{gr}_{\mathcal{F}} H^0(\mathfrak{m}, Q) \simeq \mathbb{C}[\mathcal{S}].$$

First, we set up the following:

Proposition 5.4. (i) *We have an isomorphism* $H^0(\mathfrak{m}, \operatorname{gr}_{\mathcal{F}} Q) \simeq \mathbb{C}[\mathcal{S}]$.
(ii) *For all* $i > 0$*, we have* $H^i(\mathfrak{m}, \operatorname{gr}_{\mathcal{F}} Q) = 0$.

Proof. (i) The isomorphism $\alpha : M \times \mathcal{S} \to \chi + \mathfrak{m}^\circ$ of Theorem 3.7 induces an isomorphism

$$\alpha^* : \mathbb{C}[\chi + \mathfrak{m}^\circ] \to \mathbb{C}[M] \otimes \mathbb{C}[\mathcal{S}].$$

The group M acts on $M \times \mathcal{S}$ by $x.(y, s) = (xy, s)$, with $x, y \in M$ et $s \in \mathcal{S}$, and on $\chi + \mathfrak{m}^\circ$ by the coadjoint action. This induces actions of M on $\mathbb{C}[M] \otimes \mathbb{C}[\mathcal{S}]$ and on $\mathbb{C}[\chi + \mathfrak{m}^\circ]$. We easily verify that α and α^* are M-equivariant for these actions. Thus,

$$\mathbb{C}[\chi + \mathfrak{m}^\circ]^M \simeq (\mathbb{C}[M] \otimes \mathbb{C}[\mathcal{S}])^M \simeq \mathbb{C}[M]^M \otimes \mathbb{C}[\mathcal{S}] \simeq \mathbb{C}[\mathcal{S}]$$

because $\mathbb{C}[M]^M = \mathbb{C}$. On the other hand, by Proposition 4.6, we have

$$\mathbb{C}[\chi + \mathfrak{m}^\circ]^M = \mathbb{C}[\chi + \mathfrak{m}^\circ]^{\mathfrak{m}} = H^0(\mathfrak{m}, \mathbb{C}[\chi + \mathfrak{m}^\circ]) = H^0(\mathfrak{m}, \operatorname{gr}_{\mathcal{F}} Q).$$

We have then obtained the desired isomorphism:

$$H^0(\mathfrak{m}, \operatorname{gr}_{\mathcal{F}} Q) \simeq \mathbb{C}[\mathcal{S}].$$

(ii) The isomorphisms $\vartheta : \operatorname{gr}_{\mathcal{F}} Q \to \mathbb{C}[\chi + \mathfrak{m}^\circ]$ and $\alpha^* : \mathbb{C}[\chi + \mathfrak{m}^\circ] \to \mathbb{C}[M] \otimes \mathbb{C}[\mathcal{S}]$ give, for all $i \geqslant 0$,

$$H^i(\mathfrak{m}, \operatorname{gr}_{\mathcal{F}} Q) \simeq H^i(\mathfrak{m}, \mathbb{C}[\chi + \mathfrak{m}^\circ]) \simeq H^i(\mathfrak{m}, \mathbb{C}[M] \otimes \mathbb{C}[\mathcal{S}]).$$

As a consequence of the M-action on $M \times \mathcal{S}$, we have $H^i(\mathfrak{m}, \mathbb{C}[M] \otimes \mathbb{C}[\mathcal{S}]) = H^i(\mathfrak{m}, \mathbb{C}[M]) \otimes \mathbb{C}[\mathcal{S}]$. On the other hand, it follows from [4, Theorem 10.1] (or [10, Lemma 5.1] or [8]) that $H^i(\mathfrak{m}, \mathbb{C}[M])$ is the i-th De Rham cohomology group for M. The latter is zero by [9, Ch. III, Theorem 3.7]. As a conclusion, $H^i(\mathfrak{m}, \mathbb{C}[M] \otimes \mathbb{C}[\mathcal{S}]) = H^i(\mathfrak{m}, \mathbb{C}[M]) \otimes \mathbb{C}[\mathcal{S}] = 0$, and the assertion (ii) follows. □

Given Proposition 5.4(i), we would like to show that $\operatorname{gr}_{\mathcal{F}} H^0(\mathfrak{m}, Q) \simeq H^0(\mathfrak{m}, \operatorname{gr}_{\mathcal{F}} Q)$. As in [7, Section 5], we can do this by using spectral sequences. By Lemma 5.1, we can associate to the filtration $(\mathscr{F}^k C^\bullet)_k$ the spectral sequence given by:

$$\begin{aligned}
E_m^{k,l} &:= \frac{\mathscr{F}^k C^{k+l} \cap \partial^{-1}(\mathscr{F}^{k+m} C^{k+l+1}) + \mathscr{F}^{k+1} C^{k+l}}{\partial(\mathscr{F}^{k-m+1} C^{k+l-1} \cap \partial^{-1}(\mathscr{F}^k C^{k+l})) + \mathscr{F}^{k+1} C^{k+l}};\\
E_\infty^{k,l} &:= \frac{\ker(\partial) \cap \mathscr{F}^k C^{k+l} + \mathscr{F}^{k+1} C^{k+l}}{\operatorname{im}(\partial) \cap \mathscr{F}^k C^{k+l} + \mathscr{F}^{k+1} C^{k+l}};\\
E^i &:= H^i(C) = H^i(\mathfrak{m}, Q).
\end{aligned}$$

It follows from [3, Ch. XV, Section 4] that

$$E_\infty^{k,l} \simeq \operatorname{gr}_{\mathscr{F},k} E^{k+l}. \tag{11}$$

We define a grading on $H^i(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q)$ by setting

$$H^i(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q)_k := \frac{\ker(\delta) \cap G_k^i}{\delta(G_k^{i-1})}.$$

Lemma 5.5. (i) *For all* $k, l \in \mathbb{Z}$*, we have* $E_1^{k,l} \simeq H^{k+l}(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q)_k$.
(ii) *We have* $\operatorname{gr}_{\mathscr{F}} H^0(\mathfrak{m}, Q) \simeq H^0(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q)$.

Proof. (i) We have

$$E_1^{k,l} = \frac{\mathscr{F}^k C^{k+l} \cap \partial^{-1}(\mathscr{F}^{k+1} C^{k+l+1}) + \mathscr{F}^{k+1} C^{k+l}}{\partial(\mathscr{F}^k C^{k+l-1} \cap \partial^{-1}(\mathscr{F}^k C^{k+l})) + \mathscr{F}^{k+1} C^{k+l}}.$$

Remark 5.2 and the following commutative diagram,

$$\begin{array}{ccccc}
G_k^{i-1} & \xrightarrow{\quad\delta\quad} & G_k^{i} & \xrightarrow{\quad\delta\quad} & G_k^{i+1} \\
\big\downarrow\sim & & \big\downarrow\sim & & \big\downarrow\sim \\
\mathscr{F}^k C^{i-1}/\mathscr{F}^{k+1}C^{i-1} & \xrightarrow{\partial} & \mathscr{F}^k C^{i}/\mathscr{F}^{k+1}C^{i} & \xrightarrow{\partial} & \mathscr{F}^k C^{i+1}/\mathscr{F}^{k+1}C^{i+1}
\end{array}$$

give us,

$$H^i(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q)_k \simeq \frac{\ker(\delta) \cap \mathscr{F}^k C^i/\mathscr{F}^{k+1}C^i}{\delta(\mathscr{F}^k C^{i-1}/\mathscr{F}^{k+1}C^{i-1})} \simeq \frac{\mathscr{F}^k C^i \cap \partial^{-1}(\mathscr{F}^{k+1}C^{i+1}) + \mathscr{F}^{k+1}C^i}{\partial(\mathscr{F}^k C^{i-1}) + \mathscr{F}^{k+1}C^i}.$$

In particular, when $i = k + l$, we obtain (i).

(ii) By Proposition 5.4(ii), for all $(k, l) \in \mathbb{Z}^2$ such that $k + l > 0$, we have $E_1^{k,l} \simeq H^{k+l}(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q)_k = 0$. Since $C^i = 0$ for all $i < 0$, for all $(k, l) \in \mathbb{Z}^2$ such that $k + l < 0$, we have $E_1^{k,l} = 0$.

Let $k, l \in \mathbb{Z}$ verifying $k + l = 0$ and $1 < s < \infty$. By what foregoes, we have $E_1^{u,v} = 0$ for $u + v = -1$ and $E_1^{u,v} = 0$ for $u + v = 1$. It follows from [3, Ch. XV, Proposition 5.2] that

$$E_1^{k,l} \simeq E_s^{k,l}, \text{ for all } s > 1.$$

The spectral sequence $(E_s^{k,l})_s$ is stationary. As a consequence, $E_1^{k,l} \simeq E_\infty^{k,l}$. By (i), we have $E_1^{k,l} \simeq H^{k+l}(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q)_k = H^0(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q)_k$. On the other hand, by (11), we have $E_\infty^{k,l} \simeq \operatorname{gr}_{\mathscr{F},k} H^{k+l}(\mathfrak{m}, Q) = \operatorname{gr}_{\mathscr{F},k} H^0(\mathfrak{m}, Q)$, whence (ii). ☐

We are now in the position to prove Theorem 4.7:

Proof of Theorem 4.7. By Proposition 5.4(i) and Lemma 5.5(ii), we have:

$$\mathbb{C}[\mathcal{S}] \simeq H^0(\mathfrak{m}, \operatorname{gr}_{\mathcal{F}} Q) = H^0(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q) \simeq \operatorname{gr}_{\mathscr{F}} H^0(\mathfrak{m}, Q) = \operatorname{gr}_{\mathcal{F}} H^0(\mathfrak{m}, Q) = \operatorname{gr}_{\mathcal{F}} H.$$

It remains to show that ν is an isomorphism of graded Poisson algebras. This follows from Proposition 3.8 and Remark 4.4. The graded algebra structures are clearly preserved as we saw before (at the end of Section 4). ☐

Remark 5.6. *For all $i > 0$, one has $H^i(\mathfrak{m}, Q) = 0$.*

Proof of Remark 5.6. Let $k, l \in \mathbb{Z}$ be such that $k + l > 0$. By Lemma 5.5 (proof of (ii)), we have $E_1^{k,l} = H^{k+l}(\mathfrak{m}, \operatorname{gr}_{\mathscr{F}} Q)_k = 0$. By [3, Ch. XV, Proposition 5.1], we deduce that $0 = E_\infty^{k,l} = \operatorname{gr}_{\mathscr{F},k} H^{k+l}(\mathfrak{m}, Q)$. Thus, $\operatorname{gr}_{\mathscr{F}} H^i(\mathfrak{m}, Q) = 0$ for all $i > 0$. Hence, $\mathscr{F}^k H^i(\mathfrak{m}, Q) = \mathscr{F}^{k+1} H^i(\mathfrak{m}, Q)$ for all k and all $i > 0$. For k big enough, $\mathscr{F}^k H^i(\mathfrak{m}, Q) = 0$. We deduce that $\mathscr{F}^k H^i(\mathfrak{m}, Q) = 0$ for all k and the remark follows. ☐

Let $\mathcal{C}$ be the abelian category of finitely generated left $U(\mathfrak{g})$-modules on which the element $m - \chi(m)$ acts locally nilpotently for each $m \in \mathfrak{m}$ and $\mathcal{C}'$ the category of finitely generated left H-modules.

Recall that the Skryabin's equivalence [17] sets up a category equivalence between the categories $\mathcal{C}$ and $\mathcal{C}'$ when $\mathfrak{m} = \mathfrak{m}_\chi$. This extends to our case:

Theorem 5.7. *The functor $V \mapsto Q \otimes_H V$ sets up an equivalence of the categories $\mathcal{C}'$ and $\mathcal{C}$. The inverse equivalence is given by the functor $E \mapsto \mathrm{Wh}(E)$ where $\mathrm{Wh}(E) := \{x \in E \; ; \; mx = \chi(m)x, \text{ for all } m \in \mathfrak{m}\}$.*

The same lines of arguments as used in [7, Section 6] can be applied to our situation, so we will not repeat the proof.

Theorem 5.7 raises natural questions:

$*$ Do the algebras $H(\mathfrak{g}, e, \mathfrak{m})$ depend up to isomorphism on the choice of the χ-admissible algebra $\mathfrak{m}$?

$*$ If not, what (new?) representations of $\mathfrak{g}$ do we obtain from the Skryabin's equivalence?

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